2.1.97

Exercise 2.1.97 Find the difference quotient $\frac{f(x+h)-f(x)}{h}$ where $h \neq 0$ for $f(x)=\sqrt{4-x^2}$.

Solution

We have

$$\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{4-(x+h)^2}-\sqrt{4-x^2}}{h}$$

$$= \left(\frac{\sqrt{4-(x+h)^2}-\sqrt{4-x^2}}{h}\right)\left(\frac{\sqrt{4-(x+h)^2}+\sqrt{4-x^2}}{\sqrt{4-(x+h)^2}+\sqrt{4-x^2}}\right)$$

inspired by #90 in the problems worked in the supplemental notes (to rationalize the numerator)

$$= \frac{\left(\sqrt{4-(x+h)^2}\right)^2-\left(\sqrt{4-x^2}\right)^2}{h\left(\sqrt{4-(x+h)^2}+\sqrt{4-x^2}\right)} = \frac{4-(x+h)^2-(4-x^2)}{h\left(\sqrt{4-(x+h)^2}+\sqrt{4-x^2}\right)}$$

$$= \frac{4-x^2-2xh-h^2-4+x^2}{h\left(\sqrt{4-(x+h)^2}+\sqrt{4-x^2}\right)} = \frac{h(-2x-h)}{h\left(\sqrt{4-(x+h)^2}+\sqrt{4-x^2}\right)}$$

$$= \boxed{\frac{-2x-h}{\sqrt{4-(x+h)^2}+\sqrt{4-x^2}}}$$ since $h \neq 0$. □