Exercise 3.3.85 A projectile is fired from a cliff 200 ft above the water at an inclination of 45° to the horizontal with a muzzle velocity of 50 ft/sec. The height h of the projectile above the water is modeled by

$$h(x) = \frac{-32x^2}{50^2} + x + 200$$

where x is the horizontal distance of the projectile from the face of the cliff.
(a) At what horizontal distance from the face of the cliff is the height of the projectile a maximum?
(b) Find the maximum height of the projectile?
(c) At what horizontal distance from the face of the cliff will the projectile strike the water?

Solution

(a) Now $h(x)$ is a quadratic function with $a = -32/50^2$, $b = 1$, and $c = 200$. Since $a < 0$ then h has a maximum at its vertex. The x-coordinate of the vertex is

$$\frac{-b}{2a} = \frac{-(1)}{2(-32/50^2)} = \frac{50^2}{64} = \frac{625}{16} \text{ ft.}$$

So the maximum height occurs when the distance from the cliff is $\boxed{x = 625/16 \text{ ft.}}$

This is approximately 39.0 ft.

3.3.85 continued

(b) The maximum height is the y-coordinate of the vertex and is

$$f\left(\frac{625}{16}\right) = \frac{-32\,(625/16)^2}{50^2} + \left(\frac{625}{16}\right) + 200$$

$$= \frac{-32}{(16)^2}\frac{(625)^2}{(50)^2} + \frac{625}{16} + 200 = -\frac{1}{8}\left(\frac{625}{4}\right) + \frac{625}{16} + 200$$

$$= \frac{-625}{32} + \frac{1350}{32} + \frac{6400}{32} = \boxed{\frac{7125}{32} \text{ ft}}$$

This is approximately 222.7 ft.

(c) The projectile will strike the water when the height is 0 ft. So we consider

$$h(x) = \frac{-32x^2}{50^2} + x + 200 = 0 \text{ or}$$

$$-32x^2 + 50^2 x + (50^2)(200) = -32x^2 + 2500x + 500{,}000 = 0.$$

From the quadratic formula we have

$$x = \frac{-(2500) \pm \sqrt{(2500)^2 - 4(-32)(500{,}000)}}{2(-32)}$$

$$= \frac{625}{16} \mp \frac{\sqrt{70{,}250{,}000}}{64} = \frac{625}{16} \mp \frac{500\sqrt{281}}{64}.$$

Since the physics implies that $x > 0$ when $h(x) = 0$, so we take $\boxed{x = \frac{625}{16} + \frac{125\sqrt{281}}{16} = \frac{625 + 125\sqrt{281}}{16} \text{ ft}}$

This is approximately 170.0 ft.

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