Exercise 4.4.13 Consider $P(x) = \dfrac{x^4+x^2+1}{x^2-1}$. Graph P by following Steps 1 through 7.

Solution

Step 1 Factor the numerator and denominator and find the domain. First, notice that the numerator can be interpreted as a quadratic in x^2, $(x^2)^2 + (x^2) + 1$, where $a = b = c = 1$. But there are no zeros of this polynomial function since the discriminant is $b^2 - 4ac = (1)^2 - 4(1)(1) = -3 < 0$. So the numerator is irreducible. The denominator factors as $x^2 - 1 = (x+1)(x-1)$. So

$$\boxed{P(x) = \frac{x^4+x^2+1}{x^2-1} = \frac{x^4+x^2+1}{(x+1)(x-1)}}$$

and the domain excludes -1 and 1 so that the $\boxed{\text{domain is } (-\infty,-1)\cup(-1,1)\cup(1,\infty).}$

Step 2 Put the rational function in lowest terms. Since the numerator and denominator have no common factors then P is in lowest terms as $\boxed{P(x) = \dfrac{x^4+x^2+1}{(x+1)(x-1)}}$.

Step 3 Locate the intercepts and describe the behavior at each x-intercept. For the y-intercept set $x = 0$ and we get $P(0) = \dfrac{(0)^4+(0)^2+1}{(0)^2-1} = -1$.

So the $\boxed{y\text{-intercept is } -1.}$ Since the numerator is never 0, then $y = P(x)$ is never 0 and there are $\boxed{\text{no } x\text{-intercepts.}}$

Step 4 Find the vertical asymptotes. We consider P in lowest terms and see that vertical asymptotes occur where the denominator $(x+1)(x-1)$ is 0. So the $\boxed{\text{vertical asymptotes are } x = -1 \text{ and } x = 1.}$

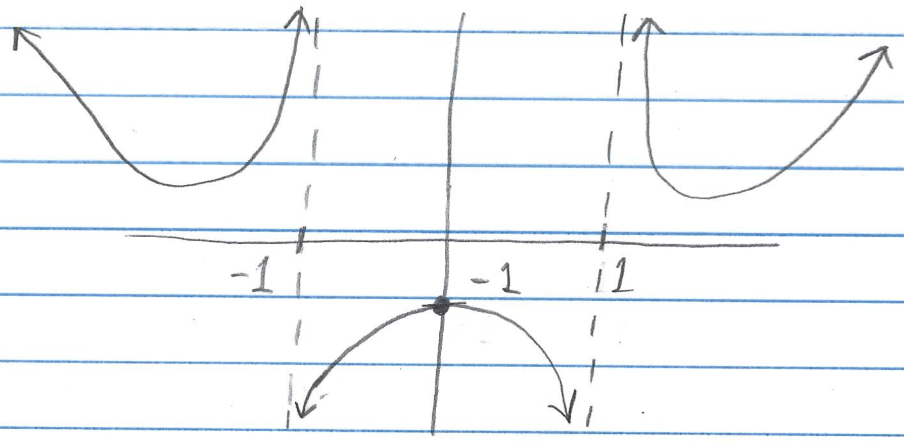
Step 5 Locate the horizontal and oblique asymptotes. Since the numerator of P is of degree $n = 4$ and the denominator is of degree $m = 2$, then $\boxed{\text{there are neither horizontal nor obique asymptotes}}$ by Note 4.2.B(4) (since $4 = n \geq m + 2 = 2 + 2 = 4$).

Step 6 Determine where the graph is above or below the x-axis. We remove the zeros of the numerator and denominator from $\mathbb{R}$ and test the sign of P on these intervals. We get the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$ and consider:

Interval	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
Test Value c	$c = -2$	$c = 0$	$c = 2$
Value of $P(c)$	7	-1	7
Conclusion	P positive	P negative	P positive

$\boxed{\text{So P is above the } x\text{-axis on } (-\infty, -1) \cup (1, \infty) \text{ and P is below the } x\text{-axis on } (-1, 1).}$

Step 7 Graph.



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