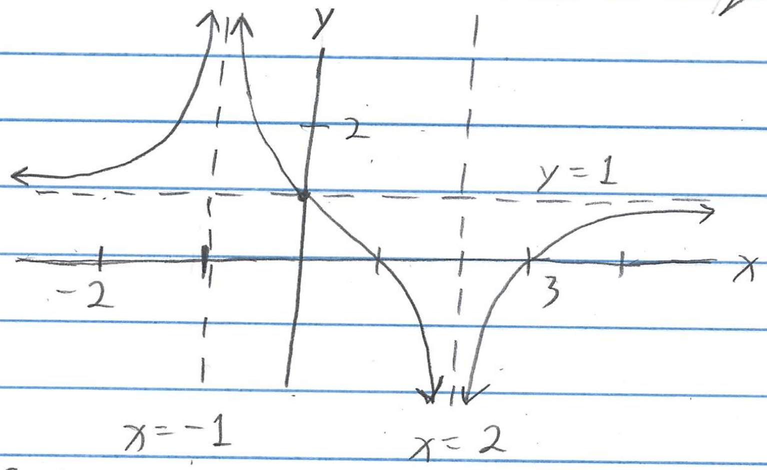
Exercise 4.4.53 Find a rational function that might have the given graph. (More than one answer may be possible.)



Solution

Let $R(x)$ be the function. There are vertical asymptotes at $x=-1$ and $x=2$ so we make $x-(-1)=x+1$ and $x-2$ factors of the denominator. Since R is positive on both sides of -1 (and near -1) then we make the factor $x+1$ of even multiplicity, say $(x+1)^2$. Since R is negative on both sides of 2 (and near 2) then we make the factor $(x-2)$ of even multiplicity, say $(x-2)^2$. Since the x-intercepts are $x=1$ and $x=3$, we make $x-1$ and $x-3$ factors of the numerator. Since the graph crosses the x-axis at 1 and 3, we make these factors of odd multiplicity.

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4.4.53 continued

Since R has a horizontal asymptote, by Note 4.2.B(2) the numerator and denominator must be of the same degree. Since the graph contains the point $(0,1)$, we need $R(0)=1$. We try $\frac{(x-1)^3(x-3)}{(x+1)^2(x-2)^2}$.

This has the correct x-intercepts, correct signs, and correct asymptotes. But at $x=0$, this function is $3/4$. So we need to scale this function at $x=0$ by $4/3$ without affecting the other properties. We can insert another factor of $x^2+4/3$ in the numerator. But this alone affects the horizontal asymptote. To keep the degrees of numerator and denominator the same we either decrease the multiplicity of $x-1$ by 2, increase the multiplicity of $x+1$ by 2, or increase the multiplicity of $x-2$ by 2 (but this will require a different scaling at $x=0$). This gives three possible answers:

$$\boxed{R_1(x)=\frac{(x-1)(x-3)(x^2+4/3)}{(x+1)^2(x-2)^2},\quad R_2(x)=\frac{(x-1)^3(x-3)(x^2+4/3)}{(x+1)^4(x-2)^2} \text{ or } R_3(x)=\frac{(x-1)^3(x-3)(x^2+16/3)}{(x+1)^2(x-2)^4}.}$$

$\square$