Exercise 4.5.57 Consider

$$f(x) = \frac{(x+4)(x^2-2x-3)}{x^2-x-6}.$$

(a) Graph $y = f(x)$, and (b) Solve $f(x) \geq 0$.

Solution

(a) We follow the 7 steps of the previous section.

Step 1 Factor the numerator and denominator and find the domain. We have

$$f(x) = \frac{(x+4)(x^2-2x-3)}{x^2-x-6} = \frac{(x+4)(x+1)(x-3)}{(x+2)(x-3)}$$

and the domain is $(-\infty,-2) \cup (-2,3) \cup (3,\infty)$.

Step 2 Put the rational function in lowest terms. Since the numerator and denominator have the common factor of $x+3$, then in lowest terms, $f(x) = \dfrac{(x+4)(x+1)}{(x+2)}$ where $x \neq 3$

Step 3 Locate the intercepts. For the y-intercept set $x=0$ and we get $f(0) = \dfrac{((0)+4)((0)+1)}{((0)+2)} = 2$

and the y-intercept is 2. The x-intercepts are -4 and -1.

Step 4 Find the vertical asymptotes. These occur when the denominator of f in lowest terms is 0. So $x=-2$ is the vertical asymptote.

Step 5 Locate the horizontal and oblique asymptotes. In lowest terms, the degree of the numerator is $n=2$ and the degree of the denominator is $m = 1$, so by Note 4.2.B(3) there is an oblique asymptote and by Note 4.2.B(2) there is no horizontal asymptote. We divide (and notice $(x+4)(x+1) = x^2+5x+4$):

$$\begin{array}{r}
x+3 \\
x+2\ \overline{)\ x^2+5x+4} \\
-(x^2+2x) \\
\hline
3x+4 \\
-(3x+6) \\
\hline
-2
\end{array}$$

so $y = x+3$ is the oblique asymptote.

Step 6 Determine where the graph is above or below the x-axis. We remove the zeros of the numerator and denominator from $\mathbb{R}$ and test the sign of f on the intervals $(-\infty, -4)$, $(-4, -2)$, $(-2, -1)$, $(-1, 3)$, $(3, \infty)$.

Interval	$(-\infty, -4)$	$(-4, -2)$	$(-2, -1)$
Test Value c	-5	-3	$-3/2$
Value of $f(c)$	$\frac{(-1)(-4)}{(-3)}$	$\frac{(1)(-2)}{(-1)}$	$\frac{(5/2)(-1/2)}{(1/2)}$
Conclusion	f negative	f positive	f negative

Interval	$(-1, 3)$	$(3, \infty)$
Test Value c	0	4
Value of $f(c)$	$\frac{(4)(1)}{(2)}$	$\frac{(8)(5)}{(6)}$
Conclusion	f positive	f positive

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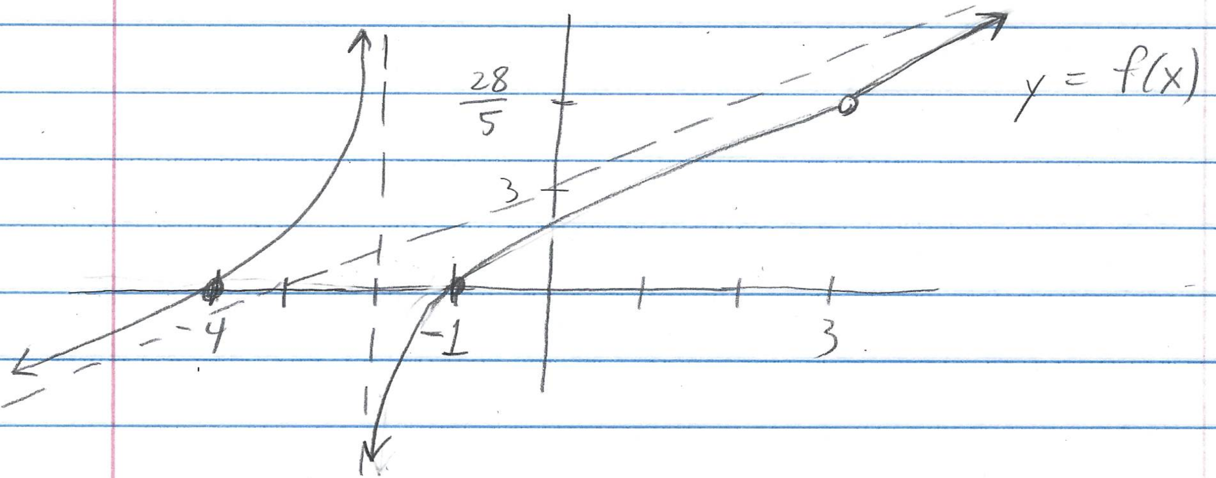
Step 7 Graph. Notice that for $x=3$,

$$\frac{(x+4)(x+1)}{x+2} = \frac{((3)+4)((3)+1)}{(3)+2} = \frac{28}{5}.$$

The point $(3, 28/5)$ is not on $y=f(x)$ but the graph "tries" to pass through this point.



(b) From the graph (and the tables above), we have $\boxed{f(x) \geq 0 \text{ on } [4,-2) \cup [-1,3) \cup (3,\infty).}$

$\square$