Exercise 5.2.65 The function $f(x) = \frac{x^2-4}{2x^2}$, $x>0$, is one-to-one. (a) Find its inverse function f^{-1} and check your answer. (b) Find the domain and the range of f and f^{-1}.

Solution

(a) We follow the 3 Steps of this section. First, set $y = f(x)$ so that $y = \frac{x^2-4}{2x^2}$, $x>0$.

Interchange the variables x and y to obtain $x = \frac{y^2-4}{2y^2}$, $y>0$. Second, we solve for y:

$$2y^2x = y^2-4,\ y>0 \text{ or } 2y^2x - y^2 = -4,\ y>0$$

$$\text{or } y^2(2x-1) = -4,\ y>0 \text{ or } y^2 = \frac{-4}{2x-1},\ y>0$$

$$\text{or } y^2 = \frac{4}{1-2x},\ y>0 \text{ or } \sqrt{y^2} = \sqrt{\frac{4}{1-2x}},\ y>0$$

$$\text{or } |y| = \sqrt{\frac{4}{1-2x}},\ y>0 \text{ or } y = \sqrt{\frac{4}{1-2x}}$$

(since $|y| = y$ because $y>0$). Therefore $f^{-1}(x) = \sqrt{\frac{4}{1-2x}}$. We check:

$$f(f^{-1}(x)) = f\left(\sqrt{\frac{4}{1-2x}}\right) = \frac{\left(\sqrt{\frac{4}{1-2x}}\right)^2 - 4}{2\left(\sqrt{\frac{4}{1-2x}}\right)^2}$$

$$= \frac{\left(\frac{4}{1-2x}\right) - 4}{2\left(\frac{4}{1-2x}\right)} \quad \text{for } x < \frac{1}{2}$$

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$$= \frac{\left(\frac{4}{1-2x}\right) - 4}{\left(\frac{8}{1-2x}\right)}\left(\frac{1-2x}{1-2x}\right) \text{ for } x < \frac{1}{2}$$

$$= \frac{4 - 4(1-2x)}{8} \text{ for } x < \frac{1}{2}$$

$$= \frac{4-4+8x}{8} \text{ for } x < \frac{1}{2}$$

$$= x \text{ for } x < \frac{1}{2},$$

and $f^{-1}(f(x)) = f^{-1}\left(\frac{x^2-4}{2x^2}\right), \quad x > 0$

$$= \sqrt{\frac{4}{1 - 2\left(\frac{x^2-4}{2x^2}\right)}}, \quad x > 0$$

$$= \sqrt{\frac{4}{1 - \frac{x^2-4}{x^2}}}, \quad x > 0$$

$$= \sqrt{\frac{4}{1 - \frac{x^2-4}{x}}\left(\frac{x^2}{x^2}\right)}, \quad x > 0$$

$$= \sqrt{\frac{4x^2}{x^2 - (x^2-4)}}, \quad x > 0$$

$$= \sqrt{\frac{4x^2}{4}}, \quad x > 0$$

$$= \sqrt{x^2} = |x| = x \quad (\text{since } |x| = x \text{ for } x > 0).$$

Third, $f(f^{-1}(x)) = x$ for all $x < \frac{1}{2}$ (that is, for all x in the domain of f^{-1}),

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and $f^{-1}(f(x)) = x$ for all $x > 0$ (that is, for all x in the domain of f). Therefore, f and f^{-1} are inverses.

(b) The domain of f is the range of g (and vice-versa), so:

the domain of f is $(0, \infty)$,
the domain of f^{-1} is $(-\infty, 1/2)$,
the range of f is $(-\infty, 1, 2)$, and
the range of f^{-1} is $(0, \infty)$.

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