Exercise A.10.67 Rationalize the denominator and assume all variables are positive:

$$\frac{\sqrt{x+h}-\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}.$$

Solution

We multiply be something appropriate to eliminate the square-roots in the denominator. We take our inspiration from the equation $(x^2-a^2)=(x-a)(x+a)$:

$$\left(\frac{\sqrt{x+h}-\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)\left(\frac{\sqrt{x+h}-\sqrt{x}}{\sqrt{x+h}-\sqrt{x}}\right)$$

$$=\frac{\left(\sqrt{x+h}-\sqrt{x}\right)^2}{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}=\frac{\left(\sqrt{x+h}-\sqrt{x}\right)^2}{(x+h)-(x)}$$

$$=\frac{\left(\sqrt{x+h}\right)^2-2\sqrt{x+h}+\left(\sqrt{x}\right)^2}{h}$$

$$=\frac{(x+h)-2\sqrt{x+h}+(x)}{h}=\boxed{\frac{2x+h-2\sqrt{x+h}}{h}}.\ \square$$