Exercise A.3.101 Factor the polynomial completely: $x^6 - 2x^3 + 1$.

Solution

We know how to factor degree 2 polynomials, so we rewrite the polynomial as

$$x^6 - 2x^3 + 1 = (x^3)^2 - 2(x^3) + 1.$$

This is of the form $y^2 + By + C$ where $y = x^3$, $B = -2$, and $C = 1$. So we search for integers a and b such that

$$ab = C = 1 \text{ and } a + b = B = -2.$$

Since $ab = 1$ then either (1) $a = b = 1$, or (2) $a = b = -1$. To get $a + b = -2$, we take $a = b = -1$. Then

$$y^2 + By + C = (y+a)(y+b)$$ or

$$(x^3)^2 - 2(x^3) + 1 = (x^3 + 1)(x^3 + 1) = (x^3 + 1)^2.$$

The Sum of Two Cubes formula states $x^3 + a^3 = (x+a)(x^2 - ax + a^2)$, so with $a = 1$ here we have $(x^3 + 1) = (x+1)(x^2 + x + 1)$. Therefore,

$$x^6 - 2x^3 + 1 = (x^3 + 1)^2 = \left((x+1)(x^2 + x + 1)\right)^2$$

$$= \boxed{(x+1)^2 (x^2 + x + 1)^2.}$$

Notice that applying the "a,b technique" we see that $(x^2 + x + 1)$ is prime. $\square$