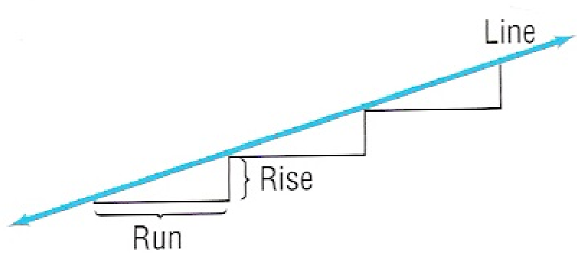
Section 1.3. Lines

Note. In this section we find intercepts and slopes of lines, graph lines given a point and the slope, find the equation for a vertical line, use the point-slope and slope-intercept formulas, graph lines, and consider parallel and perpendicular lines.

Note. If we draw a line in the xy-plane, then it will have a certain steepness which we'll describe as a ratio of the "rise" to the "run" and call the *slope* $m = \dfrac{\text{rise}}{\text{run}}$.



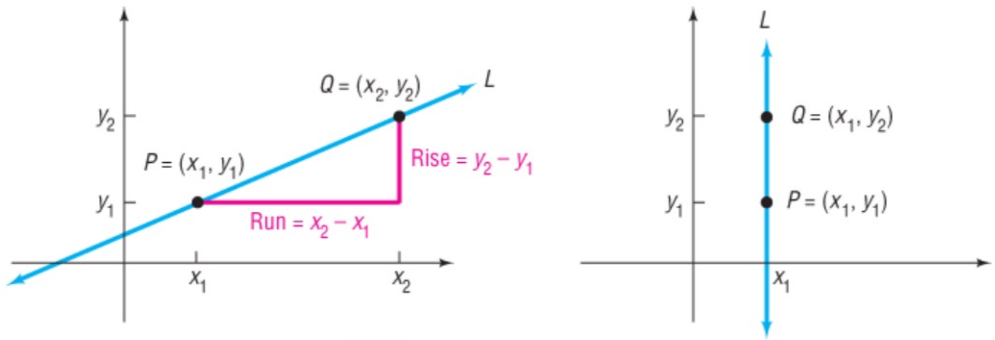
Page 19 Figure 26.

Notice that if the line is horizontal (no "rise") then the slope is $m = 0$. If $m > 0$ then the line is going "uphill" as viewed from left to right, and if $m < 0$ then the line is going "downhill" (as viewed left to right). If the line is vertical, then the slope is undefined (no "run").

Definition. Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two distinct points. If $x_1 \neq x_2$, the *slope* m of the nonvertical line L containing P and Q is defined by the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2.$$

If $x_1 = x_2$, L is a *vertical line* and the slope m of L is *undefined* (since this results in division by 0).



(a) Slope of L is $m = \frac{y_2 - y_1}{x_2 - x_1}$, $x_1 \neq x_2$ **(b)** Slope is undefined; L is vertical

Page 20 Figure 27.

Examples. Page 30 numbers 16 and 18.

Note. Sometimes the Greek letter Delta, Δ, is used to represent a change in the value of a variable. So on a line, the change in x (the "run") can be represented as $\Delta x = x_2 - x_1$ and the corresponding change in y (the "rise") can be represented as $\Delta y = y_2 - y_1$. Then the slope of a line is

$$m = \frac{\text{"rise"}}{\text{"run"}} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}.$$

More generally, if (x_1, y_1) and (x_2, y_2) are two points on the graph of a function, then the "average rate of change" of the function on the interval from x_1 to x_2 (where $x_1 < x_2$) is $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$. You will see this idea early in Calculus 1 (MATH 1910); see my Calculus 1 notes on 2.1. Rates of Change and Tangents to Curves.

Note. We also use the slope and a given point to find a second point and then graph a line.

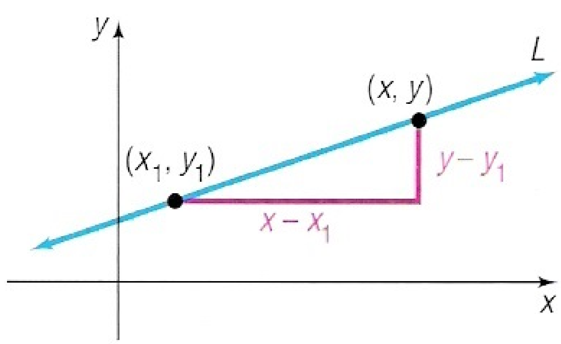
Example. Page 30 number 36.

Note. Along a vertical line, y can be any value but x does not change (no "run") and so x is constant. Therefore we have:

Theorem 1.3.A. Equation of a Vertical Line. A vertical line is given by an equation of the form $x = a$ where a is the x-intercept.

Note. Since the slope of a nonvertical line can be computed from two points on the line, suppose (x_1, y_1) is a point on the line and let (x, y) be any other point on the line (think of it as a variable point) of slope m, then we have $m = \dfrac{y - y_1}{x - x_1}$. Rearranging we get:

Theorem 1.3.B. Point-Slope Form of an Equation of a Line. An equation of a nonvertical line of slope m that contains point (x_1, y_1) is $y - y_1 = m(x - x_1)$.



Page 20 Figure 27.

Example. Page 30 number 42, Page 31 number 50.

Note. Along a horizontal line, x can be any value but y does not change (no "rise") and so y is constant. Therefore we have:

Theorem 1.3.C. Equation of a Horizontal Line. A horizontal line is given by an equation of the form $y = b$ where b is the y-intercept.

Note. From the point-slope form of a line $y - y_1 = m(x - x_1)$, we can rearrange to get $y = mx - mx_1 + y_1 = mx + b$ where $b = -mx_1 + y_1$. The point $(x_1, y_1) = (0, b)$, where b is the y-intercept, lies on the line, so we have:

Theorem 1.3.D. Slope-Intercept Form of an Equation of a Line. An equation of a line L with slope m and y-intercept b is $y = mx + b$.

Note. The slope-intercept form of a line is, in a sense, the best form. This claim will become somewhat justified when we consider functions in Chapter 2. Since this form reads "$y =$" it is also a *unique* form of the equation of a given line, unlike the point-slope form which depends on the point used in that formula.

Definition. All of the above-mentioned forms of an equation of a line can be put in the form $Ax + By = C$ for some A, B, C where not both A and B are 0. Therefore the *general form* of a line is $Ax + By = C$. This is also called a *standard form*.

Note. The formula $Ax + By = C$ can be rearranged as $y = -\frac{A}{B}x + \frac{C}{B}$, and so (if $B \neq 0$) the slope is $m = -\frac{A}{B}$ and the y-intercept is $\left(0, \frac{C}{B}\right)$. If $A \neq 0$ then the x-intercept is $\left(\frac{C}{A}, 0\right)$.

Note. Since $Ax + By = C$ can also be written as, say, $2Ax + 2By = 2C$ or $5Ax + 5By = 5C$ then the standard form of a line is not unique.

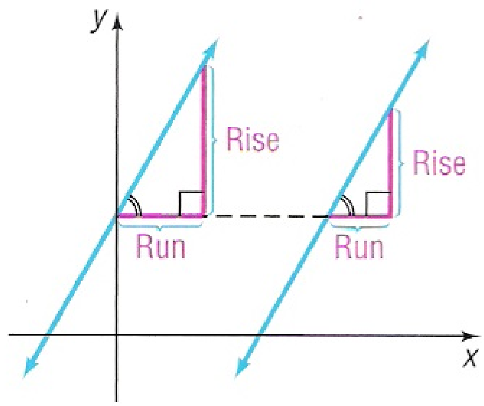
Examples. Page 31 numbers 54 and 74.

Example. Page 31 number 94.

Definition. Two lines in the same plane are *parallel* if they do not intersect.

Note. If two lines in the xy-plane are parallel, then they must have the same rate of rise/run. That is, they must have the same slope. Therefore, we have:

Theorem 1.3.E. Criterion for Parallel Lines. Two nonvertical lines are parallel if and only if their slopes are equal and they have different y-intercepts.



Page 27 Figure 43.

Example. Page 31 number 64.

Definition. Two lines which intersect at a right angle (90°) are said to be *perpendicular*.

Theorem 1.3.F. Criterion for Perpendicular Lines. Two non-vertical lines are perpendicular if and only if the product of their slopes is -1.

Note. The above theorem implies that if lines L_1 and L_2 with slopes m_1 and m_2, respectively, are perpendicular, then $m_1 \times m_2 = -1$, or $m_1 = -1/m_2$. That is, the slopes of perpendicular lines are negative reciprocals of each other.

Examples. Page 31 numbers 68 and 106.

Examples. Page 32 numbers 114 and 126.

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