Chapter 2. Functions and Their Graphs

Section 2.1. Functions

Note. In this section we find the value of a function, find the difference quotient of a function (which is used early in Calculus 1, MATH 1910; see also Chapter 14 of Sullivan's book), find the domain of a function defined by an equation, and form sums, differences, products, and quotients of two functions.

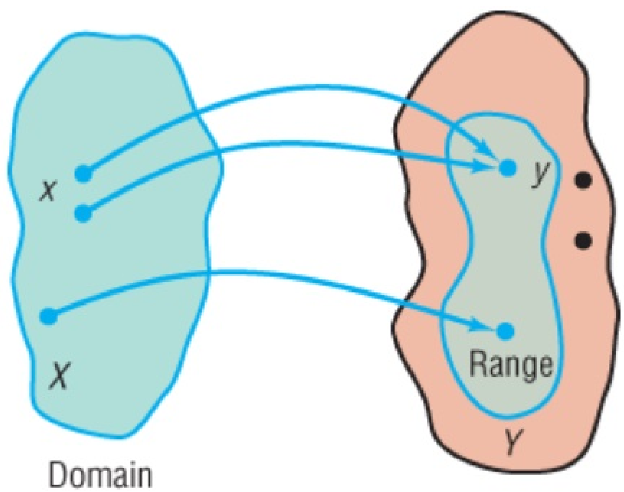
Definition. A *relation* between two sets X and Y is a set of ordered pairs of elements of the two sets. If $x \in X$, $y \in Y$, and (x, y) is in the relation, then we say x *corresponds* to y (or that y *depends on* x and sometimes write $x \to y$).

Example. We could define a relation between the people in this class and the makes of cars we drive. One element of this relation would be (Dr. Bob, Volkswagen). Another relation is probably (x, Volkswagen) where x is one of you. If so, then we can say "Dr. Bob corresponds to Volkswagen" and "x corresponds to Volkswagen," or Dr. Bob $\to$ Volkswagen and $x \to$ Volkswagen.

Notice in a relation that more than one element of X (in this example, Dr. Bob and x) may correspond to the same element of Y (in this case, Volkswagen). I also sometimes drive my wife's Toyota. So another element of the relation is (Dr. Bob, Toyota), or Dr. Bob $\to$ Toyota. So, in a relation, one element of X (in this example, Dr. Bob) may correspond to more than one element of Y (in this example, Volkswagen and Toyota).

Additional observations are that some elements of X may not correspond to anything in Y (some of you may not drive, say). Also, some elements of Y may not depend on any element of X (none of us may drive a Mercedes, say).

Definition. Let X and Y be two nonempty sets. A *function* from X to Y is a relation that associates with each element of X exactly one element of Y. The set X is the *domain* of the function. If $x \in X$, $y \in Y$, and x corresponds to y in the function, then y is the *value* of x. The *range* of the function is the set of all elements of Y which depend on elements of X.



Page 46 Figure 5.

Note. Our example about cars is a relation, but not a function. The element of X called "Dr. Bob" corresponds to more than one element of Y.

Note. The text says: "The idea behind a function is its predictability. If the input is known, we can use the function to determine the output. In 'nonfunctions' we don't have this predictability [see page 47]." Or briefly: "For a function, no input has more than one output."

Example. Page 57 Numbers 20 and 30.

Note. It's common to think of a function as a machine. You give it a good input (that is, an element of the domain) and it gives you a single output (that is, element of the range).

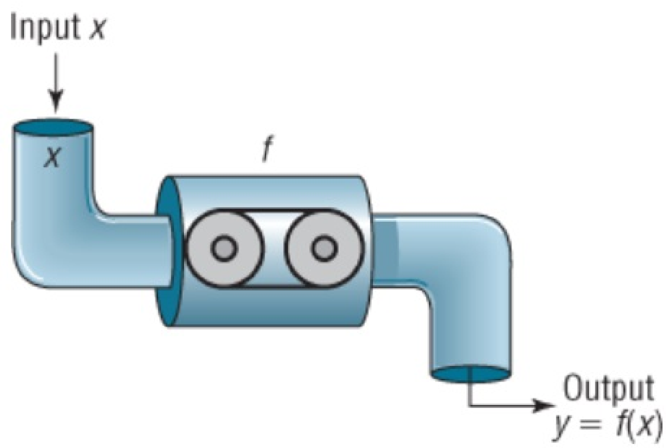
Example. The square root key on a scientific calculator represents a function. You give it a number and it outputs a (single) value. Since we cannot take square roots of negatives in here, the domain of the function is all nonnegative real numbers:

$$\{x \mid x \geq 0\} = [0, \infty).$$

What's the range?

Examples. Page 57 Numbers 34 and 36.

Note. We often denote a function with a letter. If f is a function from set X to set Y and x corresponds to y, we write $y = f(x)$, read "y equals f of x." Again, the machine concept applies and we have input x, machine f, and output y (or $f(x)$):



Page 46 Figure 5.

Definition. We abbreviate the statement "f is a function from set X into set Y" as $y = f(x)$ where x is a variable element of set X (called the *independent variable* or *argument*) and y is a variable element of set Y (called the *dependent variable* because its value depends on x).

Example. Page 57 Number 48.

Definition. When a function f is defined by an equation in x and y, we say that the function f is given *implicitly* by the equation.

Note. To see if y is a function of x, all we have to do is solve the equation for y and see if we get an expression which will represent a single y value for each "good" input value x.

Note. Not all equations in x and y define a function. WARNING: The ideas being discussed here are slightly different from the idea of a function being "implicit to an equation" as encountered in Calculus 1. See my online notes 3.7. Implicit Differentiation for details.

Examples. For the equation $xy = 4$, we can solve for y and get $y = 4/x$ so that $f(x) = 4/x$ is a function implicit to this equation. For the equation $x^2 + y^2 = 1$, we can solve for y and get $y = \pm\sqrt{1 - x^2}$ but this is not a function of x (for example, when $x = 0$ then $y = \pm 1$). However, we can substitute $y = \sqrt{1 - x^2}$ into the equation and we get an identity (this also works for $y = -\sqrt{1 - x^2}$) and $f(x) = \sqrt{1 - x^2}$ *is* a function; as mentioned above, in Calculus 1 we would say that $f(x) = \sqrt{1 - x^2}$ is implicit to the equation $x^2 + y^2 = 1$, but we don't use that terminology here since we cannot *solve* for function y in terms of x.

Note. To find the domain of a function, we need to determine which x-values are "good" values. The easiest way to do this is to find which x values are "bad," and throw them out. At this stage, the only algebraic manipulations we cannot perform are: **(1)** division by 0, and **(2)** square roots of negatives. There will be other concerns later, such as logarithms of negatives and inverse sines of numbers greater than 1, but for now these two problems are our only constraints.

Examples. Page 57 numbers 58 and 60.

Definition. The *difference quotient* of a function f at x is given by

$$\frac{f(x+h)-f(x)}{h} \text{ for } h \neq 0.$$

Examples. Page 58 numbers 80 and 90.

Definition. We can perform the following operations on two functions f and g:

(1) The *sum* $f+g$ is $(f+g)(x) = f(x) + g(x)$.

(2) The *difference* $f-g$ is $(f-g)(x) = f(x) - g(x)$.

(3) The *product* $f \cdot g$ is $(f \cdot g)(x) = f(x) \cdot g(x)$.

(4) The *quotient* $\frac{f}{g}$ is $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, when $g(x) \neq 0$.

Examples. Page 58 numbers 76.

Examples. Page 58 number 104 and Page 59 number 112.

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