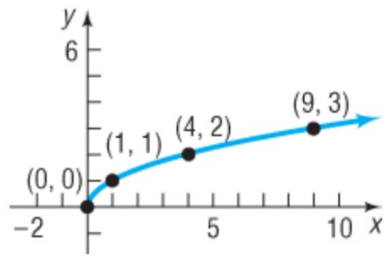
Section 2.4. Library of Functions; Piecewise-defined Functions

Note. In this section we graph the functions in the "Library of Functions" and graph piecewise-defined functions. The Library of Functions includes: $f(x) = \sqrt{x}$, $f(x) = \sqrt[3]{x}$, $f(x) = |x|$, $f(x) = b$ (b a constant), $f(x) = x$, $f(x) = x^2$, $f(x) = x^3$, $f(x) = 1/x$, and $f(x) = \text{int}(x) = \lfloor x \rfloor$.

Note. When graphing each function in the Library of Functions, we plot a few points and *trust* that the pattern continues for other points. This is a very BAD way to approach graphing, but it is the best we can do with our current knowledge. In Calculus 1 you will learn how to establish that the pattern *does* continue and see techniques to graph more general functions (see my online Calculus 1 notes for 4.3. Monotone Functions and the First Derivative Test and 4.4. Concavity and Curve Sketching). Your humble instructor has been teaching this topic off and on for 35 years and must comment that, at this level, the approach taken by Sullivan in his book is among the best available! We now go through the Library.

Note. We consider the square root function $f(x) = \sqrt{x}$. Plotting a few points we find that the graph is:



Page 83 Figure 28

Its properties include:

1. The domain and the range are the set of nonnegative real numbers.

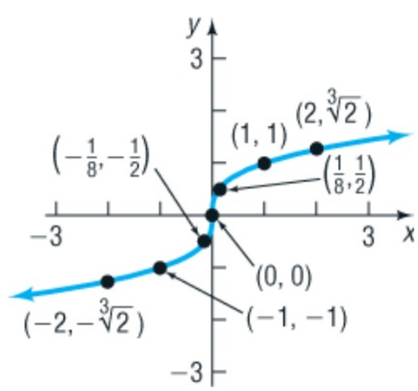
2. The x-intercept of the graph of $f(x) = \sqrt{x}$ is 0 (we now start giving only the x-coordinate of the x-intercept, and similarly for the y-intercept). The y-intercept of the graph of $f(x) = \sqrt{x}$ is also 0.

3. The function is neither even nor odd.

4. The function is increasing on the interval $(0, \infty)$.

5. The function has an absolute minimum of 0 at $x = 0$.

Note. We consider the cube root function $f(x) = \sqrt[3]{x}$. Plotting a few points we find that the graph is:



Page 84 Figure 29

Its properties include:

1. The domain and the range are the set of all real numbers.

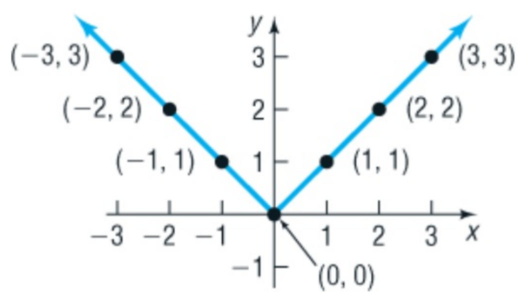
2. The x-intercept of the graph of $f(x) = \sqrt[3]{x}$ is 0. The y-intercept of the graph of $f(x) = \sqrt[3]{x}$ is also 0.

3. The function is odd. The graph is symmetric with respect to the origin.

4. The function is increasing on the interval $(-\infty, \infty)$.

5. The function does not have any local minima or any local maxima.

Note. We consider the absolute value function $f(x) = |x|$. Plotting a few points we find that the graph is:



Page 84 Figure 30

Its properties include:

1. The domain is the set of all real numbers. The range of f is $\{y \mid y \geq 0\}$.

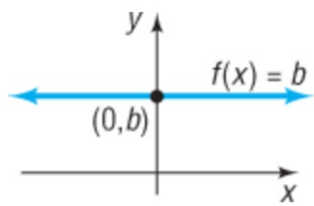
2. The x-intercept of the graph of $f(x) = |x|$ is 0. The y-intercept of the graph of $f(x) = |x|$ is also 0.

3. The function is even. The graph is symmetric with respect to the y-axis.

4. The function is decreasing on the interval $(-\infty, 0)$. It is increasing on the interval $(0, \infty)$.

5. The function has an absolute minimum of 0 at $x = 0$.

Note. We consider the constant function $f(x) = b$. Plotting a few points we find that the graph is:



Page 85 Figure 31

Its properties include:

1. The domain is the set of all real numbers. The range of f is $\{b\}$.

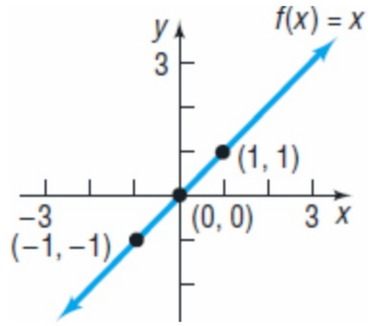
2. There is no x-intercept of the graph of $f(x) = b$ unless $b = 0$ in which case every point $(x, 0)$ is an x-intercept. The y-intercept of the graph of $f(x) = b$ is b.

3. The function is even. The graph is symmetric with respect to the y-axis.

4. The function is constant on the whole real line $(-\infty, \infty)$.

5. The function has an absolute maximum and minimum of b at each value of x.

Note. We consider the identity function $f(x) = x$. Plotting a few points we find that the graph is:



Page 85 Figure 32

Its properties include:

1. The domain is the set of all real numbers. The range of f is the set of all real numbers.

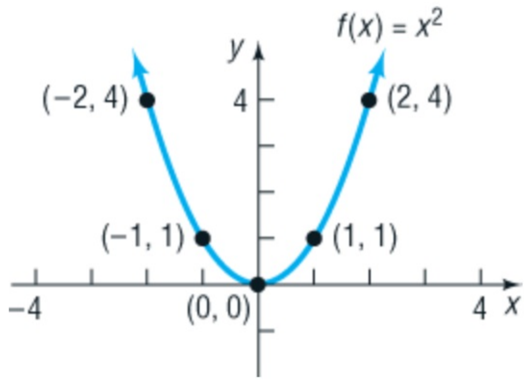
2. The x-intercept of the graph of $f(x) = x$ is 0. The y-intercept of the graph of $f(x) = x$ is also 0.

3. The function is odd. The graph is symmetric with respect to the origin.

4. The function is increasing on the interval $(-\infty, \infty)$.

5. The function has no absolute maximum nor absolute minimum.

Note. We consider the squaring function $f(x) = x^2$. Plotting a few points we find that the graph is:



Page 85 Figure 33

Its properties include:

1. The domain is the set of all real numbers. The range of f is the set of nonnegative numbers.

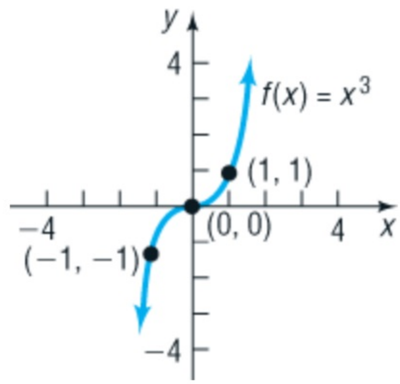
2. The x-intercept of the graph of $f(x) = x$ is 0. The y-intercept of the graph of $f(x) = x$ is also 0.

3. The function is even. The graph is symmetric with respect to the y-axis.

4. The function is decreasing on the interval $(-\infty, 0)$ and increasing on the interval $(0, \infty)$.

5. The function has an absolute minimum of 0 at $x = 0$. The function has no absolute maximum.

Note. We consider the cubing function $f(x) = x^3$. Plotting a few points we find that the graph is:



Page 86 Figure 34

Its properties include:

1. The domain is the set of all real numbers. The range of f is the set of all real numbers.

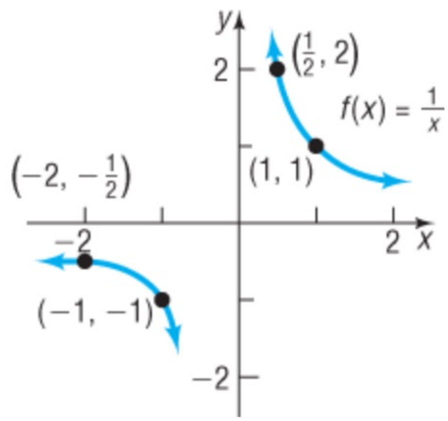
2. The x-intercept of the graph of $f(x) = x$ is 0. The y-intercept of the graph of $f(x) = x$ is also 0.

3. The function is odd. The graph is symmetric with respect to the origin.

4. The function is increasing on the interval $(-\infty, \infty)$.

5. The function has no absolute maximum nor absolute minimum.

Note. We consider the reciprocal function $f(x) = 1/x$. Plotting a few points we find that the graph is:



Page 86 Figure 37

Its properties include:

1. The domain is the set of all real numbers except 0, $(-\infty, 0) \cup (0, \infty)$. The range of f is the set of all real numbers except 0, $(-\infty, 0) \cup (0, \infty)$.

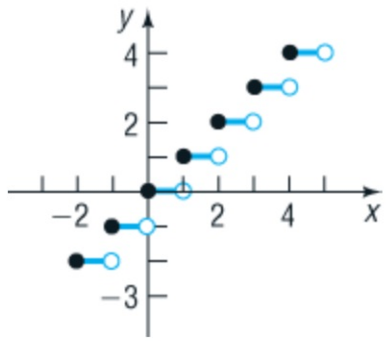
2. There is neither an x-intercept of the graph nor a y-intercept.

3. The function is odd. The graph is symmetric with respect to the origin.

4. The function is decreasing on $(-\infty, 0) \cup (0, \infty)$.

5. The function has no absolute maximum nor absolute minimum.

Note. We consider the greatest integer function $f(x) = \text{int}(x) = \lfloor x \rfloor$. This function gives out as $f(x)$ the largest integer less than or equal to x. For example, $f(n) = n$ for any integer, $f(2.5) = 2$, $f(1/2) = 0$, $f(-3/4) = -1$, and $f(-2.5) = -3$. Plotting a few points we find that the graph is:



Page 87 Figure 39

Its properties include:

1. The domain is the set of all real numbers. The range of f is the set of all integers, $\mathbb{Z}$.

2. There x-intercept of the graph are all values of x in the interval $[0, 1)$. The y-intercept is 0.

3. The function is neither even nor odd. The graph is not symmetric with respect to the x-axis, the y-axis, nor the origin.

4. The function is constant on intervals of the form $[n, n+1)$ where n is an integer.

5. The function has no absolute maximum nor absolute minimum.

Definition. When a function is defined on different parts of its domain it is called a *piecewise-defined* function.

Example. Page 88 Example 3. Consider the function

$$f(x) = \begin{cases} -2x+1 & \text{if } -3 \leq x < 1 \\ 2 & \text{if } x = 1 \\ x^2 & \text{if } x > 1. \end{cases}$$

(a) Find $f(-2)$, $f(1)$, and $f(2)$. **(b)** Determined the domain of f.

(c) Locate any intercepts. **(d)** Graph f.

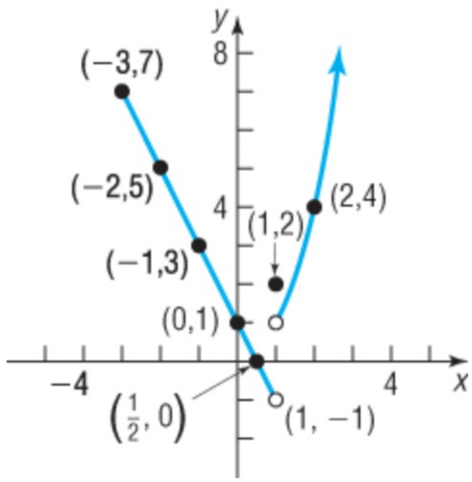
(e) Use the graph to find the range of f **(f)** Is f continuous on its domain?

Solution. (a) To find $f(-2)$, we see that $x = -2$ satisfies $-3 \leq x < 1$ so we use the piece of f defined as $-2x+1$. Hence $\boxed{f(-2) = -2(-2)+1 = 5}$. Similarly, $\boxed{f(1) = 2 \text{ and } f(2) = (2)^2 = 4}$.

(b) The domain of f is the set on which it is defined. So, in interval notation, $\boxed{\text{the domain is } [-3,1) \cup \{1\} \cup (1,\infty) = [-3,\infty)}$.

(c) For the y-intercept, we set $x = 0$ and see that $f(0) = -2(0) + 1 = 1$. So $\boxed{\text{the } y\text{-intercept is } 1}$. For the x-intercept we set each piece of f equal to 0 and see if there is a corresponding x value. First, setting $-2x+1 = 0$ we see that $x = 1/2$ and $x = 1/2$ is in the interval $[-3,1)$ where f is defined as $-2x+1$, so we have an x-intercept of $1/2$. Second, we set $2 = 0$, which is meaningless and so does not produce an x-intercept. Third, we set $x^2 = 0$ and see that $x = 0$; but $x = 0$ is *not* in the interval $(1,\infty)$ where f is defined as x^2 so this does not produce an x-intercept. Hence the only $\boxed{x\text{-intercept is } 1/2}$.

(d) We graph the line $-2x+1$ on the interval $[-3,1)$, graph the point $(x,y) = (1,2)$, and graph the squaring function x^2 (from the Library of Functions) on the interval $(1,\infty)$. This gives:



Page 89 Figure 41

(e) We see from the graph that the range of f is all real numbers greater than -1. That is, $\boxed{\text{the range is } (-1, \infty)}$.

(f) We see from the graph that the function is not continuous at $x = 1$.

NOTE: Notice from the graph that function f is only *one function*! Its name is f. However, it comes in 3 (continuous) pieces.

Example. Page 90 Number 28.

Examples. Page 91 Numbers 36 and 46.

Example. Page 92 Number 54.

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