Section 3.3. Quadratic Functions and Their Properties

Note. In this section we define and graph quadratic functions using transformations, define and identify the vertex and axis of symmetry of a quadratic function and use these to graph it, find a quadratic function from its vertex and one other point, and find maximum or minimum values of a quadratic function.

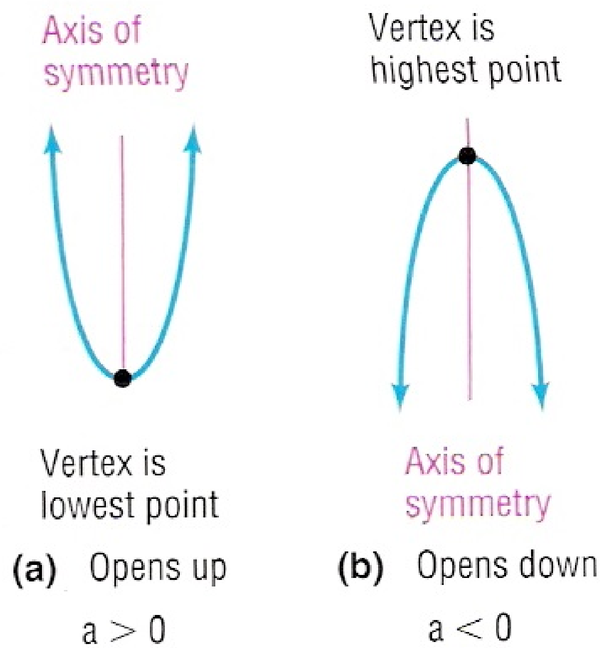
Definition. A *quadratic function* is a function of the form $f(x) = ax^2 + bx + c$ where $a \neq 0$. The domain of a quadratic function is the set of all real numbers $(-\infty, \infty) = \mathbb{R}$.

Example. A situation in which a quadratic function appears involves the motion of a projectile. Based on Newtons Second Law of Motion (force equals mass times acceleration, $F = ma$), it can be shown that, ignoring air resistance, the path of a projectile propelled upward at an inclination to the horizontal is the graph of a quadratic function.

Page 128 Figure 13

Example. Page 145 number 24.

Definition. The graph of a quadratic function $f(x) = ax^2 + bx + c$ is called a *parabola*. It *opens up* if $a > 0$ and *opens down* if $a < 0$. The lowest or highest point of a parabola is called the *vertex*. The vertical line passing through the vertex of the parabola is called the *axis of symmetry*.



Page 138 Figure 16

Note 3.3.A. We can complete the square in a quadratic function $f(x) = ax^2+bx+c$ as follows:

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c \text{ factoring out } a \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - a\left(\frac{b^2}{4a^2}\right) \text{ balancing the books} \\ &= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \text{ simplifying} \end{aligned}$$

So $f(x) = a(x-h)^2 + k$ where $h = -\dfrac{b}{2a}$ and $k = \dfrac{4ac - b^2}{4a}$. For $f(x) = ax^2 + bx + c$,

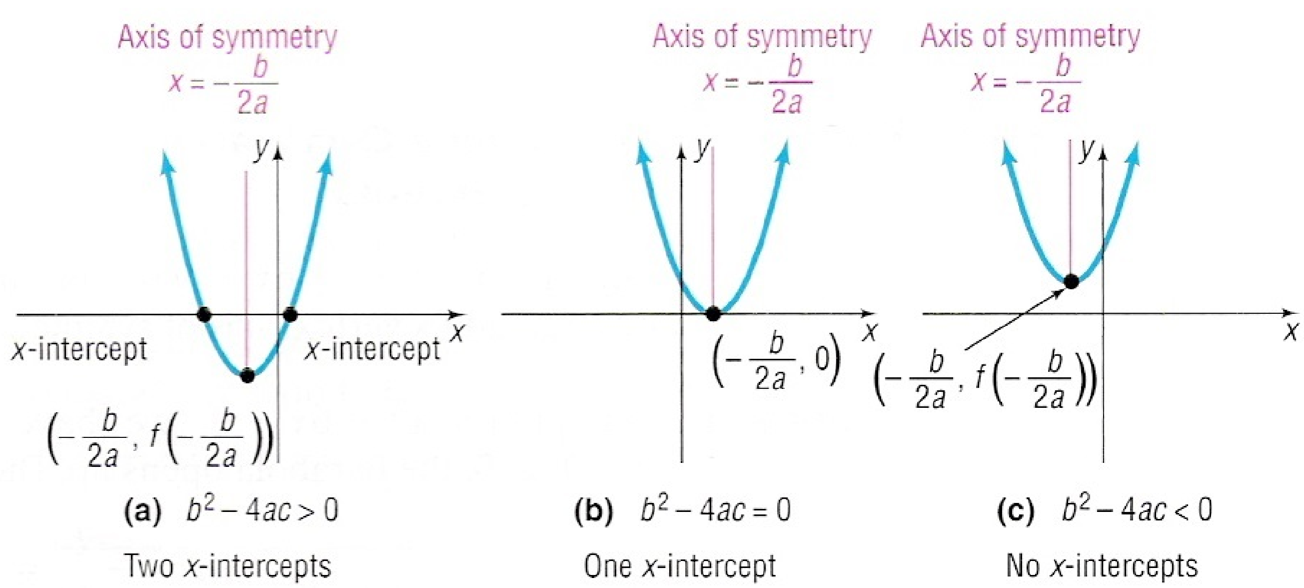
$a \neq 0$, the graph is a parabola with vertex $(h, k) = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$ and the axis of symmetry is the vertical line $x = -\frac{b}{2a}$. If $a > 0$ then the parabola opens up and the vertex is a minimum point, and if $a < 0$ then the parabola opens down and the vertex is a maximum point.

Theorem 3.3.A. The x-intercepts of a quadratic function satisfy:

(1) If the discriminant $b^2 - 4ac > 0$, the graph of $f(x) = ax^2 + bx + c$ has two distinct x-intercepts and so will cross the x-axis in two places.

(2) If the discriminant $b^2 - 4ac = 0$, the graph of $f(x) = ax^2 + bx + c$ has one x-intercept and touches the x-axis at its vertex.

(3) If the discriminant $b^2 - 4ac < 0$, the graph of $f(x) = ax^2 + bx + c$ has no x-intercept and so will not cross or touch the x-axis.



Page 141 Figure 18

Examples. Page 146 numbers 42 and 52.

Note. The quadratic function $f(x) = ax^2 + bx + c$ where $a \neq 0$ is a parabola with vertex $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$. This vertex is the highest point on the graph if $a < 0$ and the lowest point on the graph if $a > 0$. If the vertex is the highest point then $f\left(-\frac{b}{2a}\right)$ is the maximum value of f. If the vertex is the lowest point then $f\left(-\frac{b}{2a}\right)$ is the minimum value of f.

Examples. Page 146 numbers 58, 68, and 80.

Example. Page 146 number 92.

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