Chapter 4. Polynomial and Rational Functions

Section 4.1. Polynomial Functions and Models

Note. In this section we define polynomial functions and their degree, graph polynomial functions, and analyze the graph of a polynomial.

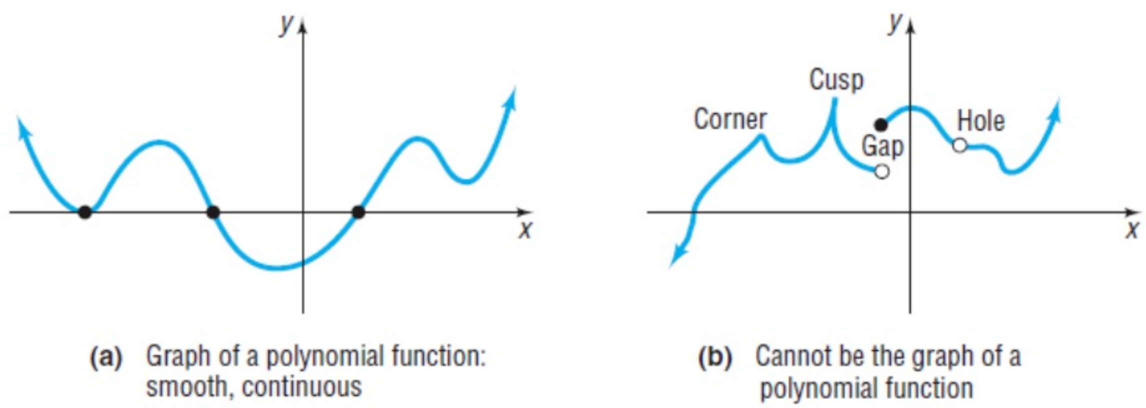
Definition. A *polynomial function* in one variable is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are constants, called *coefficients*, $n \geq 0$ is an integer, and x is the variable. If $a_n \neq 0$ then it is the *leading coefficient* and n is the *degree* of the polynomial. The monomials that sum to give the polynomial function are the *terms*. When expressed in the form above, the polynomial function is in *standard form*.

Examples. Page 184 numbers 24 and 28.

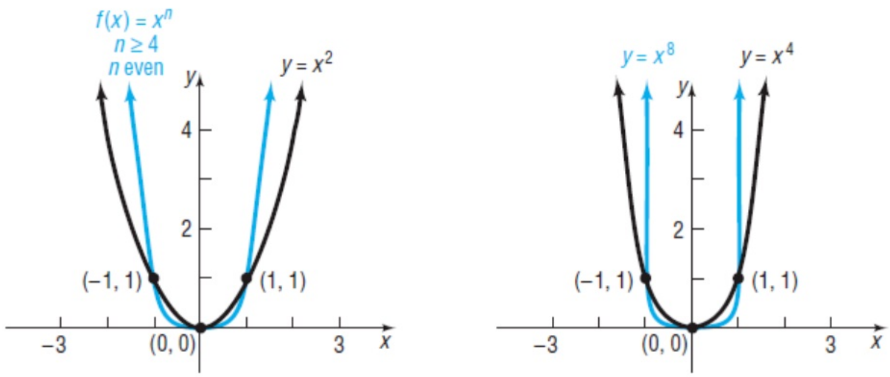
Note/Definition. In calculus you will define what it means for a function to be continuous and smooth (see my online notes for Calculus 1 on 2.5. Continuity and for Calculus 2, where "smooth" is defined, in 6.4. Arc Length). All polynomial functions, it turns out, are both smooth and continuous. For us, by *smooth* we mean that the graph contains no sharp corners or cusps; by *continuous* we mean that the graph has no gaps or holes and can be drawn without lifting your pencil from the paper. (This approach is mathematically unpleasant, but it's all we have at this stage).



Page 169 Figure 1

Definition. A *power function of degree* n is a monomial function of the form $f(x) = ax^n$ where a is a real number, $a \neq 0$, and $n > 0$ is an integer.

Note. You are familiar with the graph of $y = x^2$ from 2.4. Library of Functions. A power function $f(x) = x^n$ where n is even has a similar shape as the graph of $y = x^2$ (the shape of $y = x^2$ is a parabola, but these other functions are *not* parabolic). For $-1 < x < 1$ and $x \neq 0$, the points on the graph of $f(x) = x^n$, $n \geq 4$, are closer to the x-axis than the corresponding points on the parabola $y = x^2$. For $x < -1$ and $x > 1$, the points on the graph of $f(x) = x^n$, $n \geq 4$, are farther from the x-axis than the corresponding points on the parabola $y = x^2$.



Page 170 Figure 2

Note 4.1.A. Some properties of power functions, $f(x) = x^n$, n a positive even integer are:

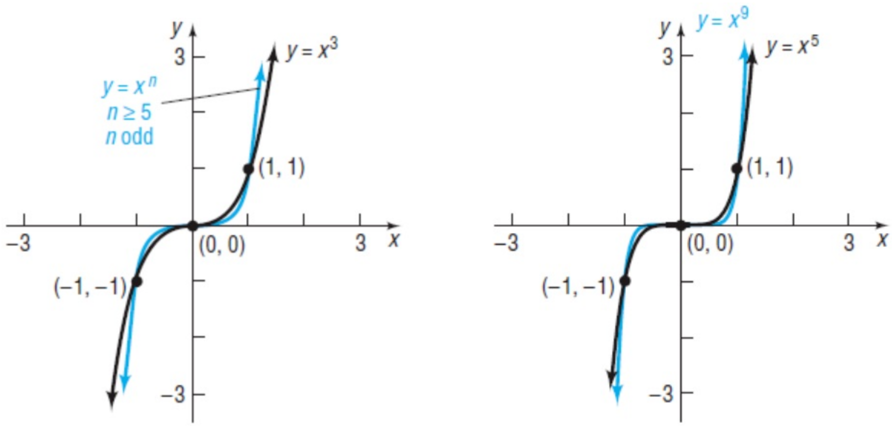
1. f is an even function, so its graph is symmetric with respect to the y-axis.

2. The domain is the set of all real numbers. The range is the set of nonnegative real numbers.

3. The graph always contains the points $(-1, 1)$, $(0, 0)$, and $(1, 1)$.

4. As the exponent n increases in magnitude, the graph is steeper when $x < -1$ or $x > 1$, but for x near the origin, the graph tends to flatten out and lie closer to the x-axis.

Note. You are familiar with the graph of $y = x^3$ from 2.4. Library of Functions. A power function $f(x) = x^n$ where n is odd is closer to the x-axis than that of $y = x^3$ if $-1 < x < 1$ and farther from the x-axis than that of $y = x^3$ if $x < 1$ or $x > 1$.



Page 171 Figures 4 and 5

Note 4.1.B. Some properties of power functions, $f(x) = x^n$, n a positive odd integer are:

1. f is an odd function, so its graph is symmetric with respect to the origin.

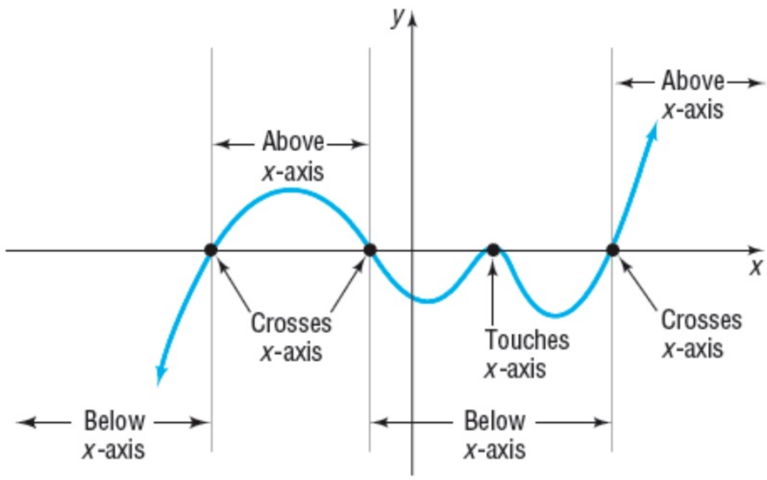
2. The domain and range are the set of all real numbers.

3. The graph always contains the points $(-1, -1)$, $(0, 0)$, and $(1, 1)$.

4. As the exponent n increases in magnitude, the graph is steeper when $x < -1$ or $x > 1$, but for x near the origin, the graph tends to flatten out and lie closer to the x-axis.

Examples. Page 184 number 38.

Note. Figure 2 below shows the graph of a polynomial function with four x-intercepts. Notice that at the x-intercepts, the graph must either cross the x-axis or touch the x-axis. Consequently, between consecutive x-intercepts the graph is either above the x-axis or below the x-axis.



Page 170 Figure 2

Definition. If f is a function and r is a real number for which $f(r) = 0$, then r is called a *real zero* of f. If f is a polynomial function, a real zero is also called a *real root*.

Examples. Page 185 numbers 46 and 52.

Definition. If $(x - r)^m$ is a factor of a polynomial f and $(x - r)^{m+1}$ is not a factor of f, then r is a *zero of multiplicity* m of f.

Example. Page 185 number 56.

Note. We can get a fair to good idea of the graph of a polynomial function by finding the x-intercepts (that is, the real zeros) and using a test value in each of the intervals that result from removing the zeros from the real line to determine if the function is positive or negative on those intervals. We illustrate this with an example.

Example 4.1.A. Consider $f(x) = 2x^2(x - 4)$. **(a)** List each real zero and its multiplicity. **(b)** Determine whether the graph crosses or touches the x-axis at each x-intercept.

Note 4.1.C. We see from the previous example that if r is a zero of even multiplicity of polynomial function f then f does not change sign from one side of r to the other. This is when we use the terminology that the graph of f *touches* the x-axis at r. If r is a zero of odd multiplicity of polynomial function f then f changes sign from one side of r to the other. This is when we use the terminology that the graph of f *crosses* the x-axis at r.

Definition/Note. The points at which a graph changes direction from increasing to decreasing or from decreasing to increasing are called *turning points*. Each turning point yields either a local maximum or a local minimum.

Note. The proof of the following requires a knowledge of Calculus 1 and the Fundamental Theorem of Algebra.

Theorem 4.1.A. If f is a polynomial function of degree n, then the graph of f has at most $n-1$ turning points. If the graph of a polynomial function f has $n-1$ turning points, then the degree of f is at least n.

Note. The behavior of the graph of a function for large values of x, either positive or negative, is referred to as its "end behavior."

Theorem 4.1.B. End Behavior. For large values of x, either positive or negative, the graph of the polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_n \neq 0$, resembles the graph of the power function $y = a_n x^n$ for large values of x, either positive or negative.

Example 4.3.B. Consider $f(x) = 2x^2(x-4)$. **(c)** Determine the maximum number of turning points on the graph. **(d)** Determine the end behavior; that is, find the power function that the graph of f resembles for large values of $|x|$.

Examples. Page 186 numbers 74 and 80.

Note 4.1.D. In summary, we analyze the graph of a polynomial function as follows:

Step 1. Determine the end behavior of the graph of the function.

Step 2. Find the x- and y-intercepts of the graph of the function.

Step 3. Determine the zeros of the function and their multiplicity. Use this information to determine whether the graph crosses or touches the x-axis at each x-intercept.

Step 4. Determine the maximum number of turning points on the graph of the function.

Step 5. Use the information in Steps 1 through 4 to draw a complete graph of the function. To help establish the y-axis scale, find additional points on the graph on each side of any x-intercept.

Example. Page 186 number 114.

Note. The text book has several very good examples. In particular, read Examples 6, 9, and 10.

Revised: 9/12/2021