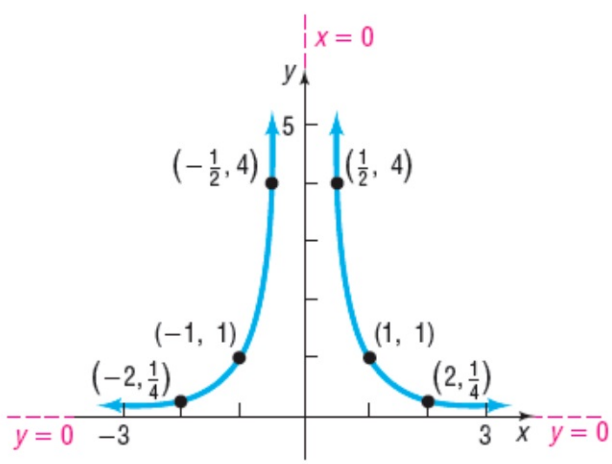
Section 4.2. Properties of Rational Functions

Note. In this section we define the domain of a rational function and find vertical/horizontal/oblique asymptotes of a rational function.

Definition. A *rational function* is a function of the form $R(x) = \frac{p(x)}{q(x)}$ where p and q are polynomial functions and q is not the zero polynomial. If p and q have no common factors, then the rational function R is said to be in *lowest terms*.

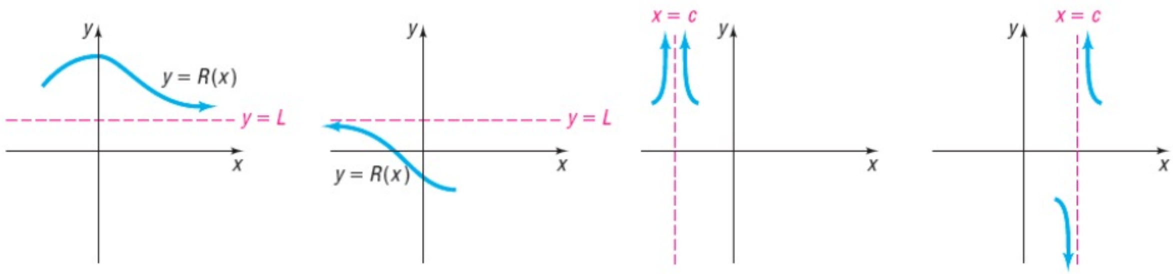
Note. The domain of a rational function is the set of all real numbers except those for which the denominator q is 0.

Note. By plotting points, using symmetry, and carefully analyzing the behavior near 0 of $H(x) = 1/x^2$, we find that it has the graph given in Figure 25 below. We can consider this function as an addition to the Library of Functions.



Page 190 Figure 25

Definition. Let R denote a function. If, as x gets arbitrarily large in either the positive or negative direction, which we denote as $x \to -\infty$ or $x \to \infty$, the values of $R(x)$ approach some fixed number L, then the line $y = L$ is a *horizontal asymptote* of the graph of R. If, as x approaches some number c, the values $|R(x)| \to \infty$ (that is, $R(x) \to -\infty$ or $R(x) \to \infty$), then the line $x = c$ is a *vertical asymptote* of the graph of R. These ideas are illustrated in Figure 27 below.



(a) End behavior: As $x \to \infty$, the values of $R(x)$ approach L [$\lim_{x\to\infty} R(x) = L$]. That is, the points on the graph of R are getting closer to the line $y = L$; $y = L$ is a horizontal asymptote.

(b) End behavior: As $x \to -\infty$, the values of $R(x)$ approach L [$\lim_{x\to-\infty} R(x) = L$]. That is, the points on the graph of R are getting closer to the line $y = L$; $y = L$ is a horizontal asymptote.

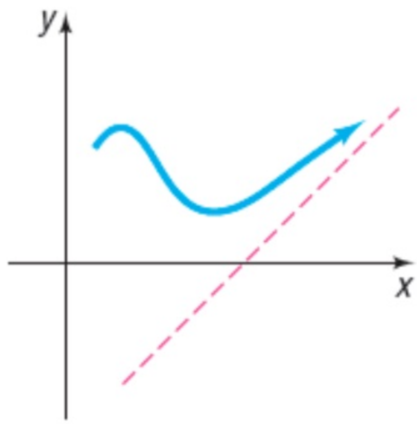
(c) As x approaches c, the values of $|R(x)| \to \infty$ [$\lim_{x\to c^-} R(x) = \infty$; $\lim_{x\to c^+} R(x) = \infty$]. That is, the points on the graph of R are getting closer to the line $x = c$; $x = c$ is a vertical asymptote.

(d) As x approaches c, the values of $|R(x)| \to \infty$ [$\lim_{x\to c^-} R(x) = -\infty$; $\lim_{x\to c^+} R(x) = \infty$]. That is, the points on the graph of R are getting closer to the line $x = c$; $x = c$ is a vertical asymptote.

Page 192 Figure 27

Note. To properly define asymptotes requires the idea of a limit from Calculus 1 (see my online notes on 2.6. Limits Involving Infinity; Asymptotes of Graphs). A common misconception is that an asymptote of a graph is something the graph "gets closer and closer to, but never gets there." This is wrong on every level! Our text book touches on this when it is stated "The graph of a function may intersect a horizontal asymptote. …The graph of a rational function will never intersect a vertical asymptote." (See page 192.) For a horizontal asymptote, an informal idea is that the graph gets (arbitrarily) close and stays close. For details, see R. Gardner, Horizontal Asymptotes: What They are Not, *The Mathematics Teacher* (Reader Reflections), February 1998, 152.

Note/Definition. There is a third possibility. If, as $x \to -\infty$ or as $x \to \infty$, the value of a rational function $R(x)$ approaches a linear expression $ax + b$, $a \neq 0$, then the line $y = ax + b$ is an *oblique* (or *slant*) *asymptote* of R. An oblique asymptote, when it occurs, describes the end behavior of the graph. The graph of a function may intersect an oblique asymptote. See Figure 28 below.



Page 192 Figure 28

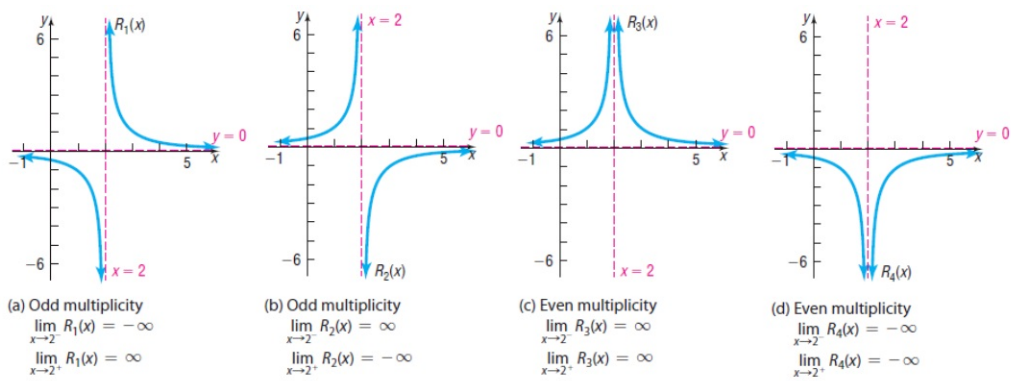
Example. Page 197 number 30

Note. We can find asymptotes largely based on the facts that (1) for x large (positive or negative), $1/x$ is small, and (2) for x positive and near 0, $1/x$ is large and positive, and for x negative and near 0, $1/x$ is large and negative.

Theorem 2.2.A. Locating Vertical Asymptotes. A rational function $R(x) = p(x)/q(x)$ in lowest terms will have a vertical asymptote $x = r$ if r is a real zero of the denominator q. That is, if $x - r$ is a factor of the denominator q of a rational function $R(x) = p(x)/q(x)$, in lowest terms, R will have the vertical asymptote $x = r$.

Examples. Page 197 number 44.

Note 4.2.A. If the multiplicity of the zero that gives rise to a vertical asymptote is odd, the graph approaches ∞ on one side of the vertical asymptote and approaches $-\infty$ on the other side. If the multiplicity of the zero that gives rise to the vertical asymptote is even, the graph approaches either ∞ or $-\infty$ on both sides of the vertical asymptote. Figure 29 illustrates several cases of this.



Page 193 Figure 29

Note 4.2.B. Consider the rational function

$$R(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_2 x^2 + b_1 x + b_0}$$

in which the degree of the numerator is n and the degree of the denominator is m.

1. If $n < m$ (the degree of the numerator is less than the degree of the denominator), the line $y = 0$ is a horizontal asymptote.

2. If $n = m$ (the degree of the numerator equals the degree of the denominator), the line $y = a_n/b_m$ is a horizontal asymptote.

3. If $n = m + 1$ (the degree of the numerator is one more than the degree of the denominator), the line $y = ax + b$ is an oblique asymptote, which is the quotient found using long division.

4. If $n \geq m+2$ (the degree of the numerator is two or more greater than the degree of the denominator), there are no horizontal or oblique asymptotes. The end behavior of the graph will resemble the power function $y = (a_n/b_m)x^{n-m}$.

Examples. Page 198 numbers 48 and 58.

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