Section 4.5. The Real Zeros of a Polynomial Function

Note. In this section we present and use the Remainder and Factor Theorems, use Descartes' Rule of Signs, use the Rational Zeros Theorem to find potential rational zeros of a polynomial function, solve polynomial equations, use the Theorem for Bounds on Zeros, and use the Intermediate Value Theorem.

Theorem 4.5.A. Division Algorithm for Polynomials. If $f(x)$ and $g(x)$ denote polynomial functions and if $g(x)$ is not the zero polynomial, then there are unique polynomial functions $q(x)$ and $r(x)$ such that

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)} \text{ or } f(x) = q(x)g(x) + r(x)$$

where $r(x)$ is either the zero polynomial or a polynomial of degree less than that of $g(x)$. $f(x)$ is the dividend, $g(x)$ is the divisor, $q(x)$ is the quotient, and $r(x)$ is the remainder.

Note. We can use the Division Algorithm (Theorem 4.5.A) to prove:

Theorem 4.5.B. Remainder Theorem. Let f be a polynomial function. If $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

Theorem 4.5.C. Factor Theorem. Let f be a polynomial function. Then $x - c$ is a factor of $f(x)$ if and only if $f(c) = 0$.

Note/Definition. The Factor Theorem tells us that finding zeros of a polynomial and finding factors of the form $x - c$ are **exactly the same problem!** Factors of the form $x - c$ are called *linear factors*.

Example. Page 233 number 18.

Note. The next theorem is related to the Fundamental Theorem of Algebra.

Theorem 4.5.D. Number of Real Zeros. A polynomial function cannot have more real zeros than its degree.

Note. In fact, as seen in the Fundamental Theorem of Algebra, the total number of both real and complex zeros (counting multiple zeros according to their multiplicity) equals the degree of the polynomial. The next theorem helps us count the number of positive and negative real zeros.

Theorem 4.5.E. Descartes' Rule of Signs. Let f denote a polynomial function written in standard form.

The number of positive zeros of f either equals the number of variations in the sign of the nonzero coefficients of $f(x)$ or else equals that number less an even integer.

The number of negative zeros of f either equals the number of variations in the sign of the nonzero coefficients of $f(-x)$ or else equals that number less an even integer.

Example. Page 233 number 30.

Note. Actually finding the zeros of a polynomial equation is an extremely difficult problem (and sometimes, in a sense, an impossible problem). The next theorem allows us to make a list of all possible rational zeros of a polynomial with integer coefficients.

Theorem 4.5.F. Rational Zeros Theorem. Let f be a polynomial function of degree 1 or higher of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0, \text{ where } a_n \neq 0, a_0 \neq 0$$

where each coefficient is an integer. If p/q, in lowest terms, is a rational zero of f, then p must be a factor of a_0, and q must be a factor of a_n.

Example. Page 233 number 42.

Note. The text book lists the following steps for finding the real zeros of a polynomial function.

Step 1. Use the degree of the polynomial to determine the maximum number of real zeros.

Step 2. Use Descartes' Rule of Signs to determine the possible number of positive zeros and negative zeros.

Step 3. **(a)** If the polynomial has integer coefficients, use the Rational Zeros Theorem to identify those rational numbers that potentially could be zeros.
(b) Use substitution, synthetic division, or long division to test each potential rational zero. Each time that a zero (and thus a factor) is found, repeat Step 3 on the depressed equation.

In attempting to find the zeros, remember to use (if possible) the factoring techniques that you already know (special products, factoring by grouping, and so on).

Example. Page 233 number 54.

Definition. A quadratic $ax^2 + bx + c$ is *irreducible* if it cannot be factored over the real numbers; that is, if it is prime over the real numbers.

Note. A quadratic of the form $x^2 + a$ where $a > 0$ has no real zeros since $x^2 + a > 0$ for all real x and hence there is no linear factor of $x^2 + a$ by the Factor Theorem, Theorem 4.5.C. That is, $x^2 + a$, $a > 0$, is irreducible. The next theorem tells us how it is possible to factor a polynomial using only real numbers.

Theorem 4.5.G. Every polynomial function (with real coefficients) can be uniquely factored into a product of linear factors and/or irreducible quadratic factors.

Note. We will prove Theorem 4.5.G in the next section using the Fundamental Theorem of Algebra. You will use this result in Calculus 2 when integrating rational functions. See my online Calculus 2 notes for 8.4. Integration of Rational Functions by Partial Fractions. As a corollary to this result we have the following, the proof of which depends on the end-behavior of an odd degree polynomial function f (for sufficiently large x, $f(x)$ and $f(-x)$ are of opposite signs).

Theorem 4.5.H. A polynomial function (with real coefficients) of odd degree has at least one real zero.

Definition. A number M is an upper bound to the zeros of a polynomial f if no zero of f is greater than M. The number m is a lower bound if no zero of f is less than m. Accordingly, if m is a lower bound and M is an upper bound to the zeros of a polynomial function f, then $m \leq$ any zero of $f \leq M$.

Note. For polynomials with integer coefficients, knowing the values of a lower bound m and an upper bound M may enable you to eliminate some potential rational zeros; that is, any potential zeros outside of the interval $[m, M]$. The next theorem gives one way to find bounds M and m.

Theorem 4.5.I. Bounds on Zeros. Let f denote a polynomial function whose leading coefficient is positive.

If $M > 0$ is a real number and $f(x) = (x - M)q(x) + R$ where the coefficients of q are nonnegative and remainder R is nonnegative, then M is an upper bound to the zeros of f.

If $m < 0$ is a real number and $f(x) = (x - m)q(x) + R$ where the coefficients of q (in standard form) followed by R alternate positive (or 0) and negative (or 0), then m is a lower bound to the zeros of f.

Example. Page 234 number 70.

Note. As observed above, actually finding the zeros of a polynomial function can be difficult/impossible. So there is a body of mathematics devoted to locating regions which contain the zeros of a polynomial. One classic example is due to Augustin Cauchy and states that the zeros of polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot + a_2 x^2 + a_1 x + a_0, \text{ where } a_n \neq 0,$$

lie in the interval $[m, M]$ where

$$m = -1 - \max\{|a_0/a_n|, |a_1/a_n|, |a_2/a_n|, \ldots, |a_{n-1}/a_n|\}$$

and

$$M = 1 + \max\{|a_0/a_n|, |a_1/a_n|, |a_2/a_n|, \ldots, |a_{n-1}/a_n|\}.$$

For this and a number of related results (which actually involve both the real and complex zeros of a polynomial) see R. Gardner and N. K. Govil, The Enestrom-Kakeya Theorem and Some of Its Generalizations, in *Current Topics in Pure and Computational Complex Analysis*, ed. S. Joshi, M. Dorff, and I. Lahiri, New Delhi: Springer-Verlag (2014), 171-200.

Note. We commented in 4.1. Polynomial Functions and Models that polynomial functions are continuous. The next result applies to continuous functions in general, but is stated here for polynomial functions. You will see this result in Calculus 1 (see my online notes for 2.5. Continuity).

Theorem 4.5.J. Intermediate Value Theorem. Let f denote a polynomial function. If $a < b$ and if $f(a)$ and $f(b)$ are of opposite sign, there is at least one real zero of f between a and b.

Example. Page 234 number 80.

Note. We can also use the Intermediate Value Theorem to numerically approximate the zeros of a polynomial function. See Page 231 Example 10.

Note. The text book describes an interesting story concerning the history of solving polynomial equations on page 232. It involves a *cubic formula*, similar to the quadratic equation. In short, the quadratic formula (which gives all solutions to the equation $ax^2+bx+c=0$) follows from completing the square and it was known by both the Babylonians and (in a limited sense) the Egyptians. The general cubic formula gives all solutions to the equation $ax^3 + bx^2 + cx + d = 0$ and was first found by Tartaglia around 1535. The general quartic formula gives all solutions to the equation $ax^4 + bx^3 + cx^2 + dx + e$ was first found by Ludovico Ferrari in 1540. Surprisingly, the mathematical world got stuck and for about 300 years failed to find a quintic formula giving the general solution to a 5th degree polynomial equation. It was shown in the early 1800s that a quintic formula does not exist (first in the work of Niels Henrik Abel in 1821 and later in a generalization by Évariste Galois in 1830). For more details, see my online notes for Introduction to Modern Algebra (MATH 4127/5127) on A Student's Question: Why The Hell Am I In This Class?

(this presentation describes the links between the classical algebra of this class and the modern algebra you might study at the senior or graduate level). In the supplement, we do 8 problems concerning the cubic formula.

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