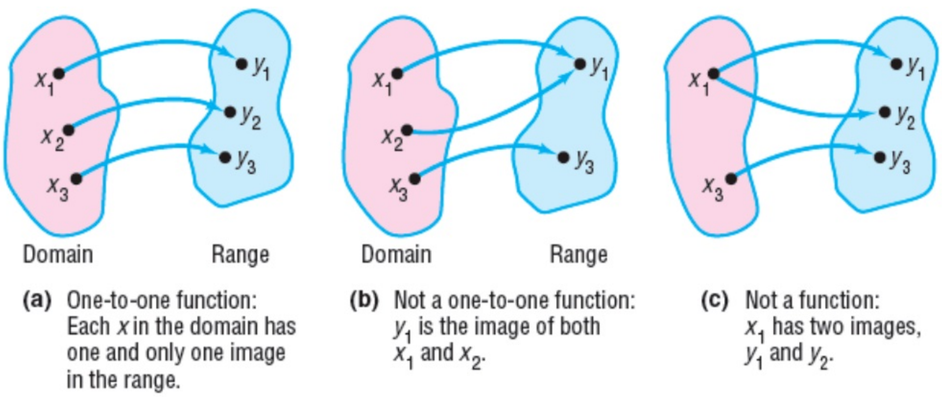
Section 5.2. One-to-One Functions; Inverse Functions

Note. In this section we determine whether a function is one-to-one, and find and graph inverse functions based on points, graphs, and equations.

Definition. A function is *one-to-one* if any two different inputs in the domain correspond to two different outputs in the range. That is, if x_1 and x_2 are two different inputs of a function f, then f is one-to-one if $f(x_1) \neq f(x_2)$.



Page 259 Figure 8

Examples. Page 265 number 14 and page 266 number 16.

Note. Just as we had a vertical line test for functions (see Theorem 2.2.A), we can test a function to see if it is one to one with a "horizontal line test." The proof of the following theorem follows directly from the definition of one-to-one.

Theorem 5.2.A. Horizontal Line Test. If every horizontal line intersects the graph of a function f in at most one point, then f is one-to-one.

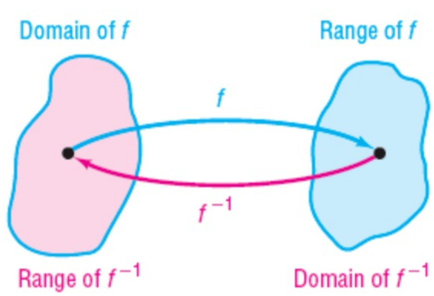
Examples. Page 266 numbers 22 and 24.

Note. We can relate the property of increasing or decreasing to the property of one-to-one as follows.

Theorem 5.2.B. A function that is increasing on an interval I is a one-to-one function on I. A function that is decreasing on an interval I is a one-to-one function on I.

Note. The next definition is the main idea of this section and the reason we introduced one-to-one functions.

Definition. Suppose that f is a one-to-one function. Then, corresponding to each x in the domain of f, there is exactly one y in the range (because f is a function), and corresponding to each y in the range of f there is exactly one x in the domain (because f is one-to-one). The correspondence from the range of f back to the domain of f is called the *inverse function* of f, denoted f^{-1}.



Page 260 Figure 11

Note. We can find inverse functions for one-to-one functions using the description of the function.

Examples. Page 266 numbers 30 and 32.

Note. Since the set of output values of f is the same as the set of input values of f^{-1} and vice-a-versa, then the domain of f is the range of f^{-1} and the domain of f^{-1} is the range of f. Also, whatever f "does" to an input value, f^{-1} "undoes" (and vice-a-versa), so we have $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}, and $f^{-1}(f(x)) = x$ for all x in the domain of f:

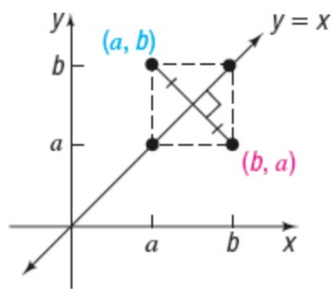
$$\boxed{\text{Input } x \text{ from domain of } f} \xrightarrow{\text{Apply } f} \boxed{f(x)} \xrightarrow{\text{Apply } f^{-1}} \boxed{f^{-1}(f(x)) = x}$$

$$\boxed{\text{Input } x \text{ from domain of } f^{-1}} \xrightarrow{\text{Apply } f^{-1}} \boxed{f^{-1}(x)} \xrightarrow{\text{Apply } f} \boxed{f(f^{-1}(x)) = x}$$

Note. The text book makes the very relevant observation that f^{-1} denotes the inverse function of f as defined above, **not** the reciprocal of f! So the little superscript in "f^{-1}" is not an exponent, but something else.

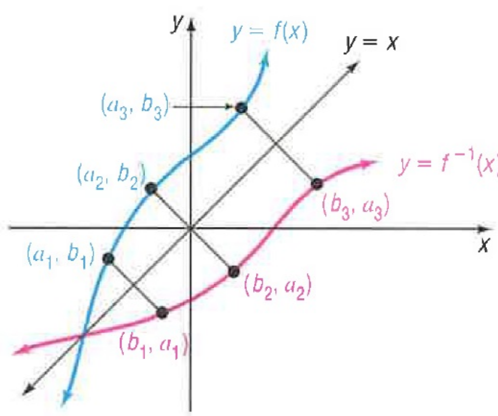
Example. Page 266 number 40 and Page 267 number 44.

Note. Suppose that (a, b) is a point on the graph of a one-to-one function f defined by $y = f(x)$. Then $b = f(a)$. This means that $a = f^{-1}(b)$, so (b, a) is a point on the graph of the inverse function f^{-1}. Geometrically, the point (b, a) on the graph of f^{-1} is the reflection about the line $y = x$ of the point (a, b) on the graph of f.



Page 262 Figure 13

Theorem 5.2.C. The graph of a one-to-one function f and the graph of its inverse function f^{-1} are symmetric with respect to the line $y = x$.



Page 262 Figure 14

Example. Page 267 number 46.

Note. To find the inverse of a one-to-one function $f(x)$, we perform these steps:

Step 1. Replace $f(x)$ with y. In $y = f(x)$, interchange the variables x and y to obtain $x = f(y)$. This equation defines the inverse function f^{-1} implicitly.

Step 2. If possible, solve the implicit equation for y in terms of x to obtain the explicit form of f^{-1}, $y = f^{-1}(x)$.

Step 3. Check the result by showing that $f^{-1}(f(x)) = f(f^{-1}(x)) = x$ for all appropriate x.

Examples. Page 267 numbers 56 and 58.

Example. Page 268 number 94

Revised: 11/11/2019