Section 5.3. Exponential Functions

Note. In this section we evaluate and graph exponential functions, define the number e, and solve exponential equations.

Note. We defined what it means to raise a real number a to a rational power r, a^r, in Appendix A.10. nth Roots; Rational Exponents. With $r = m/n$, where $m, n \in \mathbb{Z}$, $a^{m/n} = \sqrt[n]{a^m}$. We require calculus to define how to deal with irrational exponents (see my Calculus 1 notes on 3.8. Derivatives of Inverse Functions and Logarithms). The text book describes a way to estimate a^x where $a > 0$ and x is irrational; take a rational approximation r of x and then $a^x \approx a^r$, where a^r is defined above.

Theorem 5.3.A. If s, t, a, and b are real numbers with $a > 0$ and $b > 0$, then

$$a^s a^t = a^{s+t}, \ \ (a^s)^t = a^{st}, \ \ (ab)^s = a^s b^s$$

$$1^s = 1, \ \ a^{-s} = 1/a^s = (1/a)^s, \ \ a^0 = 1.$$

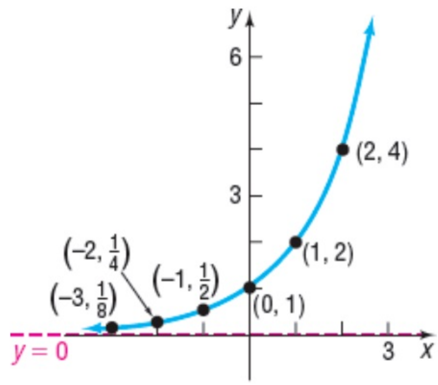
Definition/Note. An *exponential function* is a function of the form $f(x) = Ca^x$ where a is a positive real number, $a \neq 1$, and $C \neq 0$ is a real number. The domain of f is the set of all real numbers $\mathbb{R}$. The base a is the *growth factor* and $C = f(0)$ is the *initial value*.

Note. With $f(x) = Ca^x$, we have $f(x+1) = Ca^{x+1}$ so that $\dfrac{f(x+1)}{f(x)} = \dfrac{Ca^{x+1}}{Ca^x} = a$. This is expressed in the following theorem.

Theorem 5.3.B. For exponential function $f(x) = Ca^x$, $a \neq 1$, and $C \neq 0$, if x is any real number then

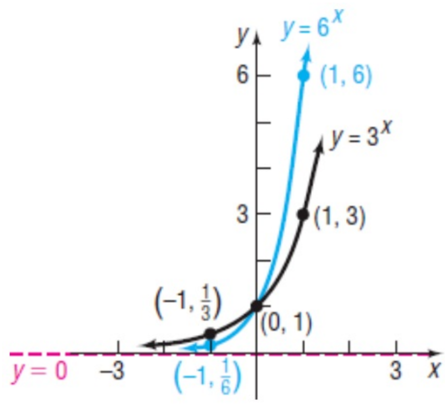
$$\frac{f(x+1)}{f(x)} = a \text{ or } f(x+1) = af(x).$$

Example 5.3.3. By plotting points, we find that the exponential function $f(x) = 2^x$ has the graph given in Figure 18 below. We can consider this function as an addition to the Library of Functions.



Page 273 Figure 18

Note. If we vary the base a in an exponential function $f(x) = a^x$ then we find that for larger values of a, the graph takes on bigger values when $x > 0$ and smaller values when $x < 0$. For example, if we compare $f(x) = 3^x$ and $g(x) = 6^x$ then we get the graphs in Figure 20:



Page 274 Figure 20

Note. When $a > 1$, the exponential function has the following properties:

1. The domain is the set of all real numbers, $(-\infty, \infty) = \mathbb{R}$, the range is the set of positive real numbers, $(0, \infty)$.

2. There are no x intercepts and the y-intercept is 1.

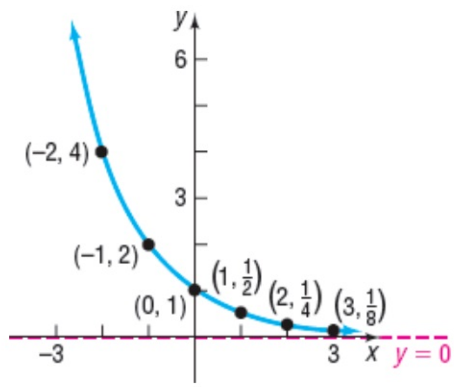
3. The x-axis, $y = 0$, is a horizontal asymptote as $x \to -\infty$.

4. $f(x) = a^x$ is an increasing and one-to-one function.

5. The graph of f contains the points $(-1, 1/a)$, $(0, 1)$, and $(1, a)$.

6. The graph of f is smooth and continuous, with no corners or gaps.

Note. Notice that if we replace x with $-x$ in the graph of $y = 2^x$, then we see that the graph of $f(x) = 2^{-x} = (2^{-1})^x = (1/2)^x$ is the reflection of the graph $y = 2^x$ about the y-axis. So the graph of $f(x) = (1/2)^x$ is as given in Figure 22. Of course, a similar behavior holds for the graph of $y = a^x$ and $f(x) = (1/a)^x$.



Page 275 Figure 22

Note. When $0 < a < 1$, the exponential function has the following properties:

1. The domain is the set of all real numbers, $(-\infty, \infty) = \mathbb{R}$, the range is the set of positive real numbers, $(0, \infty)$.

2. There are no x intercepts and the y-intercept is 1.

3. The x-axis, $y = 0$, is a horizontal asymptote as $x \to \infty$.

4. $f(x) = a^x$ is a decreasing and one-to-one function.

5. The graph of f contains the points $(-1, 1/a)$, $(0, 1)$, and $(1, a)$.

6. The graph of f is smooth and continuous, with no corners or gaps.

Examples. Page 281 numbers 38 and 52.

Note. The next definition is not well motivated at this stage. The text book states: "Many problems that occur in nature require the use of an exponential function whose base is a certain irrational number, symbolized by the letter e." See page 276. The number e is the base of the natural exponential function and the natural logarithm function. It will not be clear what is natural about these functions until you take calculus (see online Calculus 1 notes on 3.8. Derivatives of Inverse Functions and Logarithms).

Definition/Note. The real number e is defined as the number that the expression $\left(1+\dfrac{1}{n}\right)^n$ approaches as $n \to \infty$. In the notation of calculus, we have

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

We can numerically estimate e as 2.718281828459.

Note. Equations that involve terms of the form a^x, where $a > 0$ and $a \neq 1$, are called *exponential equations*. Such equations can sometimes be solved by the fact that $a^u = a^v$ implies $u = v$; this follows from the fact that exponential functions are one-to-one.

Examples. Page 282 numbers 66, 82, and 84.

Examples. Page 282 number 92 and Page 283 number 106.