Section 5.4. Logarithmic Functions

Note. In this section we change exponential statements to logarithmic statements and vice-a-versa, evaluate logarithmic expressions, find domains of and graph logarithmic functions, and solve logarithmic equations.

Note. Recall from the previous section that the exponential function $f(x) = a^x$, where $a > 0$ and $a \neq 1$, is a one-to-one function. So it has an inverse function and this inverse function is called a "logarithmic function."

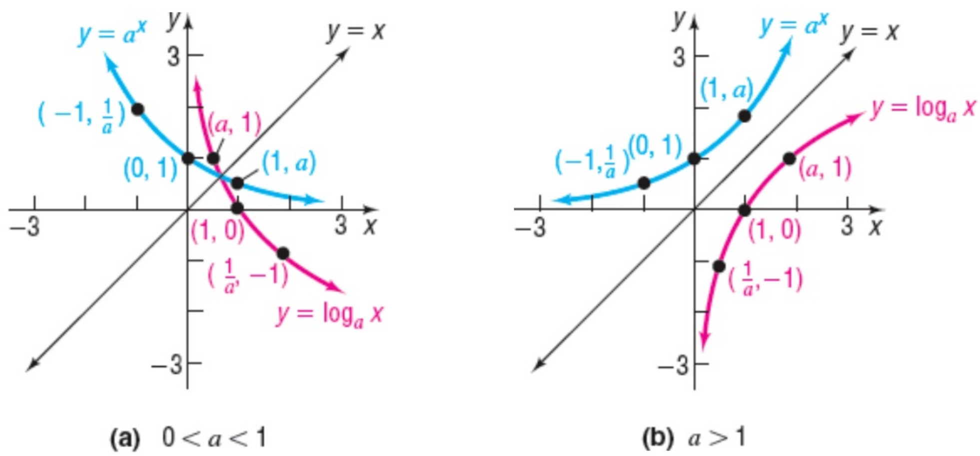
Definition. The *logarithmic function with base a*, where $a > 0$ and $a \neq 1$, is denoted $y = \log_a x$ (read "y is the logarithm with base a of x") and defined by

$$y = \log_a x \text{ if and only if } x = a^y.$$

Note. Since the range of an exponential function is $(0, \infty)$, then the domain of a logarithmic function is $x > 0$; that is $(0, \infty)$. So we cannot take the logarithm of negative numbers (if we restrict ourselves to real numbers) and we cannot take the logarithm of 0. Since the domain of an exponential function is all real numbers, $\mathbb{R} = (-\infty, \infty)$, then the range of a logarithmic function is $\mathbb{R}$. Notice that a logarithmic equation, $y = \log_a x$, is just an exponential equation rewritten, $x = a^y$. So the logarithm *is* an exponent; it is the exponent, when put on a, that gives a result of x: $a^{\log_a x} = x$.

Examples. Page 294 numbers 12, 18, 22 and Page 295 numbers 30, 36, and 48.

Note. Because exponential functions and logarithmic functions are inverses of each other, the graph of the logarithmic function $y = \log_a x$ is the reflection about the line $y = x$ of the graph of the exponential function $y = a^x$, as shown in Figure 30.



Page 288 Figure 30

Examples. Page 295 numbers 62, 66, and 70.

Note. When $a > 0$, $a \neq 1$, the logarithmic function $f(x) = \log_a x$ has the following properties:

1. The domain is the set of all positive real numbers, $(0, \infty)$, the range is the set of all real numbers, $\mathbb{R} = (-\infty, \infty)$.

2. The x intercept is 1 and there is no y-intercept.

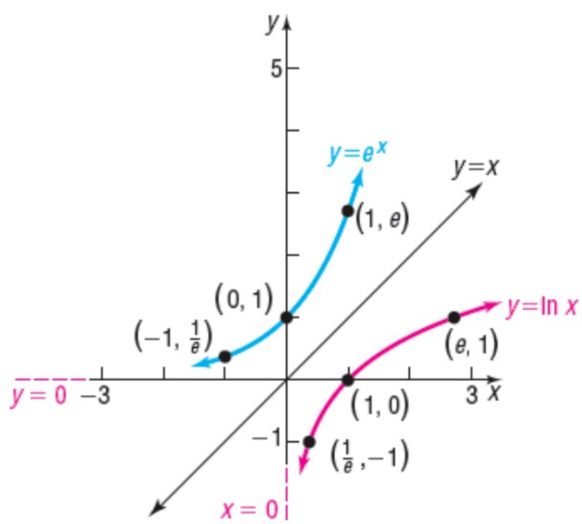
3. The y-axis, $x = 0$, is a vertical asymptote.

4. Logarithmic function $f(x) = \log_a x$ is decreasing if $0 < a < 1$ and is increasing if $a > 1$.

5. The graph of f contains the points $(1, 0)$, $(a, 1)$, and $(1/a, -1)$.

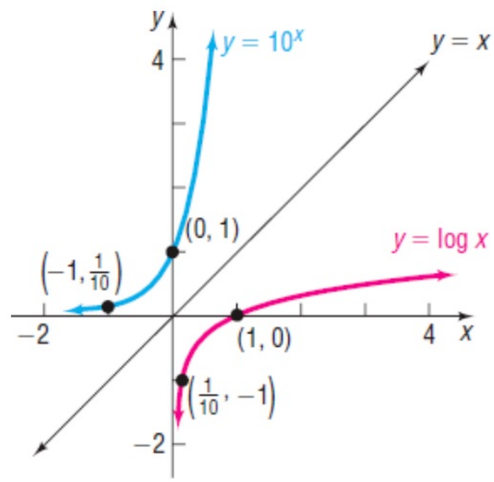
6. The graph of f is smooth and continuous, with no corners or gaps.

Note/Definition. We encountered the irrational number $e \approx 2.718281828459$ in the previous section as the base of the natural exponential function. So we define the logarithm base e as the *natural logarithm function*, denoted ln: $\ln x = \log_e x$. So we have $y = \ln x$ if and only if $x = e^y$. The graph of $y = \ln x$ and $y = e^x$ is given in Figure 33.



Page 289 Figure 33

Note/Definition. We define the logarithm base 10 as the *common logarithm function*, denoted simply as log: $\log x = \log_{10} x$. So we have $y = \log x$ if and only if $x = 10^y$. The graph of $y = \log x$ and $y = 10^x$ is given in Figure 36.



Page 290 Figure 36

Examples. Page 295 numbers 74 and 80.

Note. Equations that contain logarithms are called *logarithmic equations.* Care must be taken when solving logarithmic equations algebraically. In the expression $\log_a x = M$, remember that a and x are positive and $a \neq 1$. Some logarithmic equations can be solved by changing the logarithmic equation to exponential form using the fact that $y = \log_a x$ means $a^y = x$. Without a calculator, we will sometimes leave solutions in symbolic form.

Examples. Page 296 numbers 90, 96, 102, and 112, and Page 297 number 124.

Revised: 10/9/2021