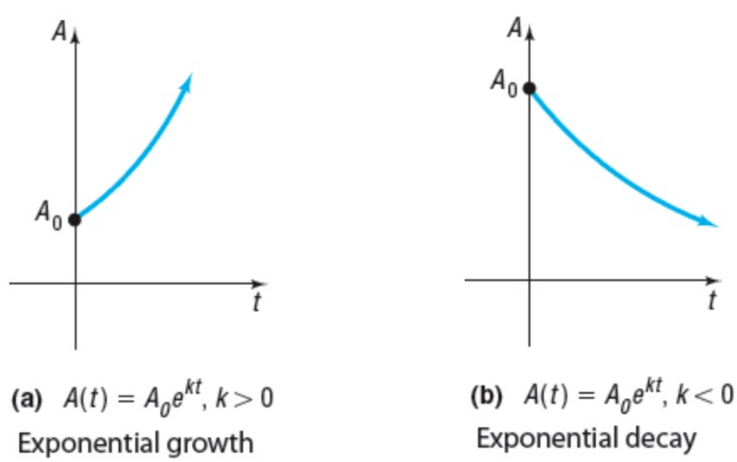
Section 5.8. Exponential Growth and Decay Models; Newton's Law; Logistic Growth and Decay Models

Note. In this section we address particular applications related to exponential functions: populations that obey the "Law of Uninhibited Growth," populations that obey the "Law of Decay," Newton's Law of Cooling, and logistic models of population growth.

Definition. Many natural phenomena have been found to follow the law that an amount A varies with time t according to the function $A(t) = A_0 e^{kt}$. Here A_0 is the original amount and $k \neq 0$ is a constant. If $k > 0$ then the amount A increases over time; if $k < 0$ then the amount A decreases over time. In either case, when an amount A varies over time according to the equation $A(t) = A_0 e^{kt}$, then it is said to follow the *exponential law*, or the *law of uninhibited growth* (when $k > 0$) or *decay* (when $k < 0$); these cases are also called simply *exponential growth* and *exponential decay*, respectively. See Figure 41.



Page 324 Figure 41

Note. Cell division is the growth process of many living organisms, such as amoebas, plants, and human skin cells. Based on an ideal situation in which no cells die and no by-products are produced, the number of cells present at a given time follows the law of uninhibited growth. However, due to the finiteness of resources, after enough time has passed growth at an exponential rate will cease and the growth rate will decrease (this is explored in the Logistic Equation below). You are likely to encounter these ideas in Differential Equations (MATH 2120); see my online notes for Differential Equations on 3.3. Rate Problems. In fact, exponential growth, exponential decay, and Newton's Law of Cooling are each addressed in Calculus 2; see 7.2. Exponential Change and Separable Differential Equations.

Example. Page 332 number 2.

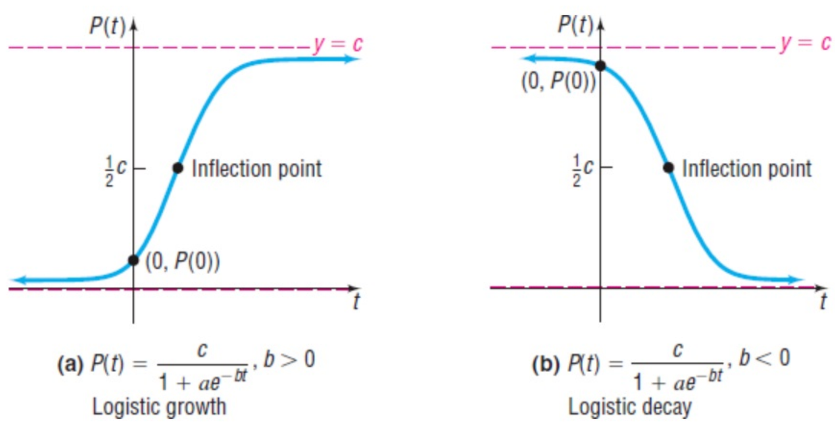
Note/Definition. The amount A of a radioactive material present at time t is given by $A(t) = A_0 e^{kt}$ where $k < 0$ and A_0 is the original amount of radioactive material. Parameter k is a negative number that represents the rate of decay. All radioactive substances have a specific *half-life*, which is the time required for half of the radioactive substance to decay. Carbon dating uses the fact that all living organisms contain two kinds of carbon, carbon-12 (a stable carbon) and carbon-14 (a radioactive carbon with a half-life of 5730 years). While an organism is living, the ratio of carbon-12 to carbon-14 is constant. But when an organism dies, the original amount of carbon-12 present remains unchanged, whereas the amount of carbon-14 begins to decrease. This change in the amount of carbon-14 present relative to the amount of carbon-12 present makes it possible to calculate when the organism died.

Example. Page 332 number 4.

Note. Newton's Law of Cooling states that the temperature u of a heated object at a given time t can be modeled by the function $u(t) = T + (u_0 - T)e^{kt}$ where $k < 0$, T is the constant temperature of the surrounding medium, and u_0 is the initial temperature of the heated object.

Example. Page 332 number 14.

Note/Definition. In a *logistic model*, the population P after time t is given by the function $P(t) = \dfrac{c}{1 + ae^{-bt}}$ where a, b, and c are constants with $a > 0$ and $c > 0$. The model is a growth model if $b > 0$; the model is a decay model if $b < 0$. The number c is called the *carrying capacity* (for growth models) because the value $P(t)$ approaches c as t approaches infinity; that is, $\lim_{t\to\infty} P(t) = c$. The number $|b|$ is the *growth rate* for $b > 0$ and the *decay rate* for $b < 0$.



Page 329 Figure 42

Note. Some properties of the logistic model are:

1. The domain is the set of all real numbers $\mathbb{R} = (-\infty, \infty)$. The range is the interval $(0, c)$, where c is the carrying capacity.

2. There are no x-intercepts and the y-intercept is $P(0)$.

3. There are two horizontal asymptotes, $y = 0$ and $y = c$.

4. $P(t)$ is an increasing function if $b > 0$ and a decreasing function if $b < 0$.

5. There is an *inflection point* where $P(t)$ equals $1/2$ of the carrying capacity. The inflection point is the point on the graph where the graph changes from being curved upward to being curved downward for growth functions, and the point where the graph changes from being curved downward to being curved upward for decay functions.

6. The graph is smooth and continuous, with no corners or gaps.

This model is addressed in Calculus 2, see 9.4. Graphical Solutions of Autonomous Equations.

Example. Page 333 number 26.

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