Appendix A. Review.

A.1. Algebra Essentials

Note. In this appendix, we consider sets, graph inequalities, define distance on the real line, evaluate algebraic expressions, use the law of exponents, and evaluate square roots.

Definition. A *set* is a well-defined collection of distinct objects. The objects of the set are its *elements*. If a set has no elements, it is called the *empty set*, denoted $\varnothing$. We often list the elements of a set using *set bracket* notation (this is called the *roster method* of presenting a set).

Example. The set of *digits* in a base 10 number system is $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Example/Definition. In *set-builder notation*, we describe the elements of a set by imposing a constraint on the elements of the set. For example,

$$D = \{x \mid x \text{ is an even digit}\}$$

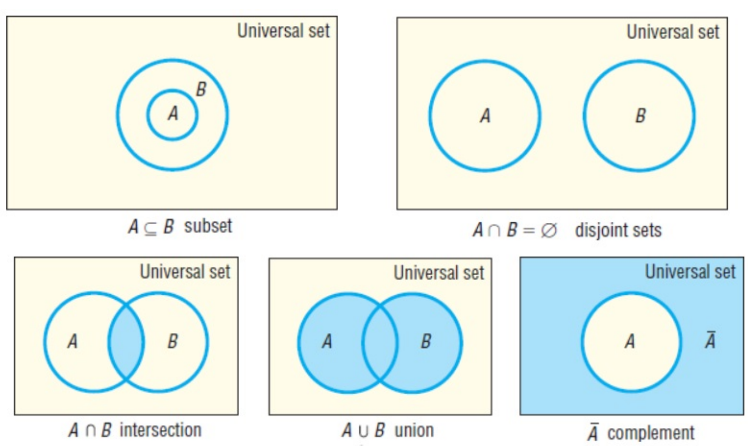
is the set of all x *such that* (as represented by the little vertical line) x is an even digit. So $D = \{0, 2, 4, 6, 8\}$.

Definition. If every element of set A is also an element of set B, then A is a *subset* of B, denoted $A \subseteq B$ or $B \supseteq A$. If two sets A and B have the same elements then A *equals* B, denoted $A = B$. The *intersection* of sets A and B, denoted $A \cap B$, is the set consisting of elements that belong to both A and B. The *union* of A and B, denoted $A \cup B$, is the set consisting of elements that belong to either A or B, or both. Sets A and B are *disjoint* if A and B have no elements in common; that is, if $A \cap B = \varnothing$.

Definition. The set consisting of all elements to be considered in a particular setting is the *universal set*, often denoted U. If A is a set, the *complement* of A, denoted $\overline{A}$ (often denoted A^c), is the set consisting of all elements in the universal set that are not in A.

Examples. Page A11 numbers 12, 14, 18, 22.

Note. It is common to illustrate relationships between sets using "Venn diagrams." We have the following diagrams.



Page A3 Figures 2 and 3

Note A.1.A. The definition of the real numbers is well beyond the scope of this course. For the record, the *real numbers* are a complete ordered field (for details, see my online notes for Analysis 1 [MATH 4217/5217] on 1-2. Properties of the Real Numbers as an Ordered Field and 1-3. The Completeness Axiom). We denote the real numbers as $\mathbb{R}$. The *natural numbers* is the set $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$. The *integers* is the set $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. The *rational numbers* is the set of all quotients of integers where the denominator is not 0, $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z}, q \neq 0\}$. The set of real numbers which are not rational is the set of *irrational numbers*. The set of rational numbers consists of all "fractions." Examples of irrational numbers include square roots of natural numbers which are not perfect squares, such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc. Also, π is irrational and the number e (which you will encounter in 5.3. Exponential Functions in this class) is irrational.

Note. Rational real numbers have decimal representations that either terminate or are nonterminating with repeating blocks of digits. Irrational real numbers have decimal representations that neither repeat nor terminate.

Note. Two properties of the real numbers are:

Distributive Property. For any real numbers a, b, c we have $a(b + c) = ab + ac$.

Zero-Product Property. For real numbers a and b, if $ab = 0$ then either $a = 0$ or $b = 0$ or both equal 0.

Note. The real numbers can be represented by points on a line called the real number line. There is a one-to-one correspondence between real numbers and points on a line. That is, every real number corresponds to a point on the line, and each point on the line has a unique real number associated with it.

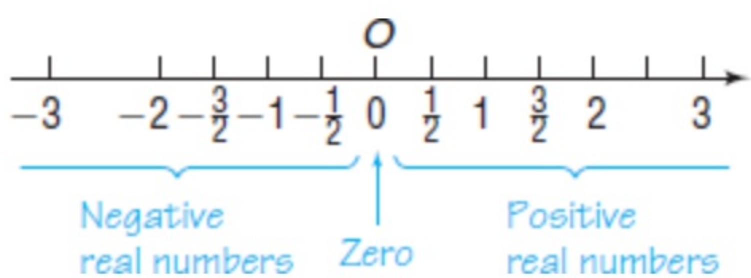
Definition. The real number associated with a point P is called the *coordinate* of P, and the line whose points have been assigned coordinates is called the *real number line.*

Note A.1.B. In drawing the real number line, we have the following conventions.

1. The negative real numbers are the coordinates of points to the left of the origin O.

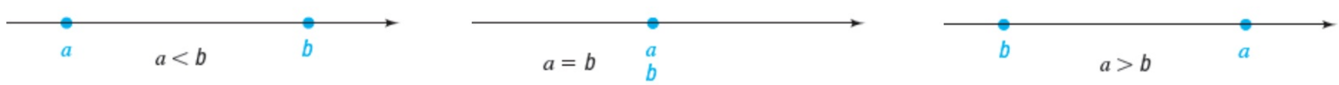
2. The real number zero is the coordinate of the origin O.

3. The positive real numbers are the coordinates of points to the right of the origin O.



Page A4 Figure 6

Note/Definition. A property of the real number line follows from the fact that, given two numbers (points) a and b, either a is to the left of b, or a is at the same location as b, or a is to the right of b. If a is to the left of b, then "a is less than b," which is written $a < b$. If a is to the right of b, then "a is greater than b," which is written $a > b$. If a is at the same location as b, then $a = b$. If a is either less than or equal to b, then $a \leq b$. Similarly, $a \geq b$ means that a is either greater than or equal to b. On the number line we have the following.



Page A5 Figure 7

Definition. An *inequality* is a statement in which two expressions are related by an inequality symbol. The expressions are referred to as the *sides* of the inequality. Inequalities of the form $a < b$ or $b > a$ are called *strict inequalities*, whereas inequalities of the form $a \leq b$ or $b \geq a$ are called *nonstrict inequalities*.

Note A.1.C. Notice that $a > 0$ is equivalent to the statement that a is positive, and $a < 0$ is equivalent to the statement that a is negative. The inequality $a \geq 0$ is read as "a is nonnegative." The inequality $b \leq 0$ is read as "b is nonpositive."

Examples. Page A11 numbers 42 and 44.

Definition. The *absolute value* of a real number a, denoted $|a|$, is defined by the rules $|a| = a$ if $a \geq 0$, and $|a| = -a$ if $a < 0$.

Definition. If P and Q are two points on a real number line with coordinates a and b, respectively, the *distance between P and Q*, denoted $d(P, Q)$, is $d(P, Q) = |b - a|$.

Examples. Page A11 numbers 46, 48, 50.

Note/Definition. We use letters such as x, y, a, b, and c to represent numbers. If the letter used is to represent any number from a given set of numbers, it is called a *variable*. A *constant* is either a fixed number or a letter that represents a fixed number. Constants and variables are combined using the operations of addition, subtraction, multiplication, and division to form *algebraic expressions*.

Examples. Page A11 numbers 52 and 54, Page A12 numbers 64 and 68.

Definition. The set of values that a variable may assume is called the *domain* of the variable.

Note A.1.D. To find the domain of a variable, we often eliminate "bad" values of the variable. The main algebraic constraints in this class are the facts that we cannot take square roots of negatives and division by 0 is undefined.

Examples. Page A12 numbers 74 and 80.

Definition. If a is a real number and n is a positive integer, then the symbol a^n represents the product of n factors of a. That is, $a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$. In the expression a^n, a is called the *base* and n is called the *exponent*, or *power*.

Definition. If $a \neq 0$, then $a^0 = 1$. If $a \neq 0$ and if n is a positive integer, then $a^{-n} = 1/a^n$.

Theorem A.1.A. Laws of Exponents. For m and n positive integers we have

$$a^m a^n = a^{m+n}, \ \ (a^m)^n = a^{mn}, \ \ (ab)^n = a^n b^n,$$
$$\frac{a^m}{a^n} = a^{m-n} = \frac{1}{a^{n-m}} \text{ if } a \neq 0, \ \ \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \text{ if } b \neq 0.$$

Examples. Page A12 numbers 90, 92, and 98.

Definition. If a is a nonnegative real number, the nonnegative number b such that $b^2 = a$ is the *principal square root* of a, denoted $b = \sqrt{a}$.

Note A.1.E. Consider the expression $\sqrt{a^2}$. Since $a^2 \geq 0$, the principal square root of a^2 is defined whether $a > 0$ or $a < 0$. Since the principal square root is nonnegative, we need an absolute value to ensure the nonnegative result. That is, $\sqrt{a^2} = |a|$ for any real number a.

Note. Consider $\sqrt{9}$. Since $3^2 = 9$ and $3 \geq 0$, then $\sqrt{9} = 3$. It is also true that $(-3)^2 = 9$, but -3 is not nonnegative so it is not the square root of 9 (this is the reason that the term *principal* square root is used; arguably -3 is another square root but it is not the principal square root). BEWARE that the square root symbol always represents a nonnegative number. If we are interested in both the positive and negative square root of 9, then we write $\pm\sqrt{9}$ and we have $\pm\sqrt{9} = \pm 3$.

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