A.3. Polynomials

Note. In this appendix, we define monomials, polynomials, take "special products" of polynomials, divide polynomials using long division, factor polynomials, and complete the square.

Definition. A *monomial* in one variable is the product of a constant and a variable raised to a nonnegative integer power. A monomial is of the form ax^k where a is a constant, x is a variable, and $k \geq 0$ is an integer. The constant a is called the *coefficient* of the monomial. If $a \neq 0$, then k is called the *degree* of the monomial. The sum or difference of two monomials having different degrees is called a *binomial*. The sum or difference of three monomials with three different degrees is called a *trinomial*.

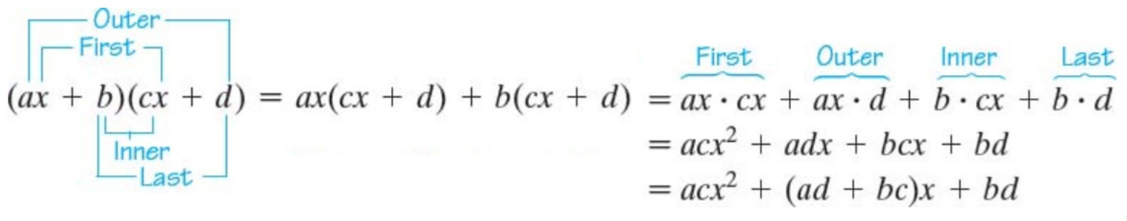
Definition. A *polynomial* in one variable is an algebraic expression of the form

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

where $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are constants, called *coefficients* of the polynomial, $n \geq 0$ is an integer, and x is a variable. If $a_n \neq 0$, then a_n is called the *leading coefficient*, a_nx^n is the *leading term*, and n is the *degree* of the polynomial. The monomials that make up a polynomial are called its *terms*. If all of the coefficients are 0, the polynomial is called the *zero polynomial*, which has no degree. (WARNING: In some settings, the zero polynomial is not considered a "polynomial.") Polynomials are usually written in *standard form*, beginning with the nonzero term of highest degree and continuing with terms in descending order according to degree (as given above).

Examples. Page A30 number 24.

Note. You might recall the **FOIL** method from high school. It involves the product of two binomials and can be illustrated as follows.



Examples. Page A30 numbers 48 and 50.

Note. The use of FOIL allows us to calculate the following "special product":

Difference of Two Squares. $(x-a)(x+a) = x^2 - a^2$.

Squares of Binomials, or Perfect Squares. $(x+a)^2 = x^2+2ax+a^2$, $(x-a)^2 = x^2 - 2ax + a^2$.

Cubes of Binomials, or Perfect Cubes. $(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$, $(x-a)^3 = x^3 - 3ax^2 + 3a^2x - a^3$.

Difference of Two Cubes. $(x-a)(x^2+ax+a^2) = x^3 - a^3$.

Sum of Two Cubes. $(x+a)(x^2-ax+a^2) = x^3 + a^3$.

Note. Consider the equation $\dfrac{842}{15} = 56 + \dfrac{2}{15}$. On the left hand side we have divided 842 by 15. The number 842 is called the *dividend* and the number 15 is called the *divisor*. On the right hand side we have expressed the fraction 842/15 as an integer plus a smaller fraction. The number 56 is called the *quotient* and the number 2 (in the numerator of the right hand side) is called the *remainder*. Notice that we can rewrite the equation as $(56)(15) + 2 = 842$ so that we have:

$$\text{(Quotient)(Divisor)} + \text{Remainder} = \text{Dividend}.$$

The division of polynomials can be dealt with similar to the long division of numbers. We illustrate this with two examples.

Examples. Page A30 numbers 66 and 74.

Note. The previous examples suggest the following general result.

Theorem A.3.A. Let Q be a polynomial of positive degree, and let P be a polynomial whose degree is greater than or equal to the degree of Q. The remainder after dividing P by Q is either the zero polynomial or a polynomial whose degree is less than the degree of the divisor Q.

Definition. Expressing a given polynomial as a product of other polynomials is called *factoring*. The other polynomials are called *factors* of the given polynomial.

Note/Definition. In this section, we will consider only polynomials with integer coefficients. The factoring problem is then called *factoring over the integers*.

Example. We can multiply out to verify that $(2x+3)(x-4) = 2x^2 - 5x - 12$. So the factors of $2x^2 - 5x - 12$ are $2x + 3$ and $x - 4$.

Definition. A polynomial with integer coefficients is *prime* if the only factors of the polynomial are 1, -1, the polynomial itself, and the negative of the polynomial. A polynomial that has been written as a product consisting only of prime factors is *factored completely*.

Note. We can use the special products above to factor certain polynomials.

Examples. Page A30 numbers 78.

Note. We will see in the not-to-distant future that we can factor any second degree polynomial $ax^2 + bx + c$. For now, we concentrate on second degree polynomials of the form $x^2 + Bx + C$.

Note. To factor $x^2 + Bx + C$ where B and C are integers, by FOIL, we try to find integers a and b whose product is C and whose sum is B. That is, find integers a and b where $B = a+b$ and $C = ab$. Then $x^2+Bx+C = (x+a)(x+b) = x^2+(a+b)x+ab$.

Examples. Page A30 numbers 83.

Note. Until the last section of Chapter 4 (Section 4.6), we will only study real numbers $\mathbb{R}$ (as opposed to complex numbers $\mathbb{C}$). Whenever a real number a is squared, the result is nonnegative: $a^2 \geq 0$ for all $a \in \mathbb{R}$. Therefore we have when restricting ourselves to real numbers:

Theorem A.3.B. Any polynomial of the form $x^2 + a^2$, a real, is prime.

Note. One can show that every polynomial with real coefficients can be factored into prime factors which consist of prime first degree polynomials and prime second degree polynomials with real coefficients (see the last theorem in Section 4.6).

Note. We can also factor polynomials "by grouping." In this method we recognize common factors and take advantage of the distributive law of multiplication over addition.

Examples. Page A30 number 124.

Note. Now we tackle the task of factoring the second degree polynomial

$$Ax^2 + Bx + C \text{ where } A \neq 1.$$

We follow these steps:

1. Find the value of AC.

2. Find integers with product AC that add up to B. That is, find integers ab such that $AC = ab$ and $B = a + b$.

3. Write $Ax^2 + Bx + C = Ax^2 + ax + bx + C$.

4. Factor by grouping.

Examples. Page A30 numbers 98 and 100.

Note. The idea behind *completing the square* is to take an expression of the form $x^2 + bx$ and add just the right thing to make it a perfect square. Notice that

$$x^2 + 2ax + a^2 = (x + a)^2 \text{ and } x^2 - 2ax + a^2 = (x - a)^2.$$

Now if we have the expression $x^2 + bx$ then treating b as $\pm 2a$ we see that we need to add $a^2 = (\pm b/2)^2 = (b/2)^2$. The two previous equations then become

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2 \text{ and } x^2 - bx + \left(\frac{b}{2}\right)^2 = \left(x - \frac{b}{2}\right)^2.$$

Example. Page A31 number 126.

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