A.6. Solving Equation

Note. In this appendix, we solve equations by factoring, solve equations involving absolute value, and solve quadratic equations by factoring/completing the square/using the quadratic formula.

Definition. An *equation in one variable* is a statement in which two expressions, at least one containing the variable, are equal. The expressions are called the *sides* of the equation. Since an equation is a statement, it may be true or false, depending on the value of the variable. Unless otherwise restricted, the admissible values of the variable are those in the domain of the variable. These admissible values of the variable, if any, that result in a true statement are called *solutions*, or *roots*, of the equation. To *solve an equation* means to find all the solutions of the equation. The *solution set* is the set which contains all solutions of the equation. If an equation is satisfied for every value of the variable for which both sides are defined, then the equation is an *identity*. Equations having the same solution set are called *equivalent equations*.

Note A.6.A. Some procedures that result in equivalent equations are:

1. Interchanging the two sides of the equation;

2. Simplifying the sides of the equation by combining like terms, eliminating parentheses, and so on;

3. Adding or subtracting the same expression on both sides of the equation;

4. Multiplying or dividing both sides of the equation by the same nonzero expression;

5. If one side of the equation is 0 and the other side can be factored, then we use the Zero-Product Property and set each factor equal to 0. (See the online notes for Appendix A.1. Algebra Essentials.)

Note A.6.B/Definition. A WARNING: Squaring both sides of an equation does not necessarily lead to an equivalent equation. For example, $x = 3$ has one solution, but squaring both sides gives $x^2 = 9$ which has two solutions, as we see by factoring, $x^2 - 9 = (x-3)(x+3) = 0$, and using the Zero-Product Property, $x = -3$ or $x = 3$. In such an event, the value $x = -3$ is called an *extraneous root*. Whenever both sides of an equation are squared (or, more generally, raised to an even power), the resulting solutions must be checked because of this behavior.

Examples. Page A51 numbers 16, 24, and 28.

Examples. Page A51 numbers 36 and 48. These examples illustrate solving equations by factoring.

Examples. Page A51 numbers 54 and 64. These examples illustrate solving equations with absolute value.

Definition. A *quadratic equation* is an equation equivalent to the form $ax^2 + bx + c = 0$ where a, b, and c are real numbers and $a \neq 0$. A quadratic equation in this form $ax^2 + bx + c = 0$ is in *standard form.* A quadratic equation is called a *second-degree equation* because, when it is in standard form, the left side is a polynomial of degree 2.

Examples. Page A51 numbers 74, 86, and Page A52 number 92.

Theorem A.6.A. The Quadratic Formula. Consider the quadratic equation $ax^2 + bx + c = 0$ where $a \neq 0$. If $b^2 - 4ac < 0$ then this equation has no real solution. If $b^2 - 4ac \geq 0$ then the real solution(s) of this equation is (are) given by the *quadratic formula*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The quantity $b^2 - 4ac$ is called the *discriminant.*

Note A.6.C. We can summarize the solutions of a quadratic equation $ax^2+bx+c = 0$ in terms of the discriminant as follows:

1. If $b^2 - 4ac > 0$ then there are two unequal real solutions.

2. If $b^2 - 4ac = 0$ then there is one "repeated solution" or a "double root."

3. If $b^2 - 4ac < 0$ then there is no real solution.

Note. It turns out that if we use complex numbers then every quadratic equation has two solutions (counting a double root twice, as needed). This follows from the Fundamental Theorem of Algebra, which is stated in Section 4.6. Complex Zeroes; Fundamental Theorem of Algebra.

Examples. Page A52 numbers 98, 104, and 110.

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