A.7. Complex Numbers; Quadratic Equations in the Complex Number System

Note. In this appendix, we define, add, subtract, multiply and divide complex numbers. We also solve quadratic equations involving complex numbers.

Note. There is no real number such that $x^2 = -1$. We now postulate the existence of a "number" (but not a real number) satisfying the property that when multiplied by itself gives -1. It turns out that this is a very mathematically fruitful thing to do! We'll explore some of the implications in 4.6. Complex Zeroes; Fundamental Theorem of Algebra. For a deeper exploration, see my online notes for Complex Variables (MATH 4337/5337).

Definition/Postulate. The *imaginary unit*, denoted i, is the number whose square is -1. That is, $i^2 = -1$. The number system that results from introducing the number i is called the *complex number system*, denoted $\mathbb{C}$.

Note. Surprisingly, it doesn't end here. Just as the real number system is embedded in the complex number system, the complex number system is embedded in the "quaternions"; for a deep mathematical adventure, see my online PowerPoint presentation on The Quaternions: An Algebraic Approach.

Definition. *Complex numbers* are numbers of the form $a + bi$, where a and b are real numbers. The real number a is called the *real part* of the number $a + bi$; the real number b is called the *imaginary part* of $a + bi$; and i is the imaginary unit, so $i^2 = -1$. When a complex number is written in the form $a + bi$, where a and b are real numbers, it is in *standard form.*

Definition. Equality, addition, subtraction, and multiplication of complex numbers are defined so as to preserve the familiar rules of algebra for real numbers. We have the following definitions.

• Two complex numbers are *equal* if and only if their real parts are equal and their imaginary parts are equal: $a + bi = c + di$ if and only if $a = c$ and $b = d$.

• Two complex numbers are *added* by forming the complex number whose real part is the sum of the real parts and whose imaginary part is the sum of the imaginary parts: $(a + bi) + (c + di) = (a + c) + (b + d)i$.

• Two complex numbers are *subtracted* by forming the complex number whose real part is the difference of the real parts and whose imaginary part is the difference of the imaginary parts (in the same order as the difference of the complex numbers): $(a + bi) - (c + di) = a + bi - c - di = (a - c) + (b - d)i$.

• The *product* of two complex numbers is formed as if FOIL applied when the complex numbers are represented in standard form:

$$(a+bi)\times(c+di) = ac+adi+bci+bdi^2 = ac+adi+bci+bd(-1) = (ac-bd)+(ad+bc)i.$$

Examples. Page A60 numbers 12, 14, and 22.

Definition. If $z = a + bi$ is a complex number, then its *conjugate*, denoted $\overline{z}$, is defined as $\overline{z} = \overline{a+bi} = a - bi$.

Note. For $z = a + bi$, we have $\overline{z} = a - bi$ and so the product $z\overline{z}$ is

$$z\overline{z} = (a+bi)(a-bi) = a^2 - abi + abi - b^2 i^2 = a^2 + b^2.$$

That is, $z\overline{z}$ is a nonnegative real number (in fact, $z\overline{z} = 0$ if and only if $z = 0 = 0 + 0i$). It turns out that this property can be used to define an "absolute value" on the complex numbers (it is often called the *modulus* of a complex number z, denoted $|z|$). It is defined as $|z| = \sqrt{z\overline{z}}$.

Theorem A.7.A. The following are properties of the conjugate operation:

(a) The conjugate of a real number is the real number itself: $\overline{a} = a$ for $a \in \mathbb{R}$.

(b) The conjugate of the conjugate of a complex number is the complex number itself: $\overline{(\overline{z})} = z$.

(c) The conjugate of the sum of two complex numbers equals the sum of their conjugates: $\overline{z+w} = \overline{z} + \overline{w}$.

(d) The conjugate of the product of two complex numbers equals the product of their conjugates: $\overline{zw} = \overline{z}\,\overline{w}$.

Examples. Page A61 number 92: Prove Theorem A.7.A(d).

Note. The fact that $z\overline{z}$ is real is useful in dealing with division by complex numbers and in finding the reciprocal of a complex number, as we now illustrate.

Examples. Page A60 numbers 26 and 30.

Note. Similar to the fact that the number -1 oscillates between the values of 1 and -1 as it is raised to powers, the number i also only takes on a few values as it is raised to powers:

$i^1 = i$	$i^2 = -1$	$i^3 = (i^2)i$ $= (-1)i = -i$	$i^4 = (i^3)i = (-i)(i)$ $= -i^2 = -(-1) = 1$
$i^5 = (i^4)i$ $= (1)i = i$	$i^6 = (i^4)(i^2)$ $= (1)(-1) = -1$	$i^7 = (i^4)(i^3)$ $= (1)(-i) = -i$	$i^8 = (i^4)(i^4)$ $= (1)(1) = 1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$i^{4n+1} = (i^{4n})i$ $= (1)^n i = i$	$i^{4n+2} = (i^{4n})(i^2)$ $= (1)^n(-1) = -1$	$i^{4n+3} = (i^{4n})(i^3)$ $= (1)^n(-i) = -i$	$i^{4n+4} = (i^{4n})(i^4)$ $= (1)^n(1) = 1$

Examples. Page A61 numbers 36, 38, and 48.

Definition. If N is a positive real number, we define the *principal square root* of $-N$, denoted by $\sqrt{-N}$, as $\sqrt{-N} = \sqrt{N}i$ where i is the imaginary unit and $i^2 = -1$.

Note. The complex valued principal square root function does not satisfy all of the same properties of the real valued principal square root function. For example, the complex valued principal square root of a product does not equal the product of the principal square roots, as we illustrate with an example. Consider $10 = \sqrt{100} = \sqrt{(-25)(-4)}$ and $\sqrt{-25}\sqrt{-4} = \sqrt{25}i\sqrt{4}i = (5i)(2i) = 10i^2 = -10$. So we see that $\sqrt{(-25)(-4)} \neq \sqrt{-25}\sqrt{-4}$.

Theorem A.7.B. Quadratic Formula. In the complex numbers system, the solutions of the quadratic equation $ax^2 + bx + c = 0$ where a, b, and c are real numbers and $a \neq 0$, are given by the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Note. In the complex number system, consider a quadratic equation $ax^2+bx+c = 0$ where a, b, and c are real numbers and $a \neq 0$.

1. If $b^2 - 4ac > 0$, the equation has two unequal real solutions.

2. If $b^2 - 4ac = 0$, the equation has a repeated real solution, a double root.

3. If $b^2 - 4ac < 0$, the equation has two complex solutions that are not real. The solutions are conjugates of each other.

Examples. Page A61 numbers 60, 70, and 78.

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