Chapter 3. Polynomial and Rational Functions

3.3. Rational Functions I

Note. In preparation for this section, you may need to review Appendix A Sections A.3 and A.4, Section 1.2, and Section 2.5.

Definition. A *rational function* is a function of the form $R(x) = \dfrac{p(x)}{q(x)}$ where p and q are polynomial functions and q is not the zero polynomial. The domain is the set of all real numbers except those for which the denominator q is 0.

Example. Page 171 number 18.

Definition. If $R(x) = \dfrac{p(x)}{q(x)}$ is a rational function and if p and q have no common factors, then the rational function R is said to be in *lowest terms*. For a rational function $R(x) = \dfrac{p(x)}{q(x)}$ in lowest terms, the zeros, if any, of the numerator are the x-intercepts of the graph of R and so will play a major role in the graph of R.

Note. The graph of $H(x) = \dfrac{1}{x^2}$ is:

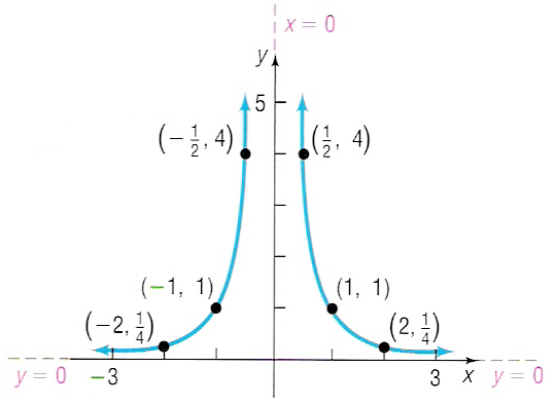

Figure 38, Page 164

Since $H(x)$ gets arbitrarily large when x is sufficiently close to 0, we say that "the limit as x approaches 0 of $H(x)$ is ∞" and write $\lim_{x \to 0} H(x) = \infty$. Since $H(x)$ gets arbitrarily close to 0 as x gets large, we say that "the limit as x approaches ∞ of $H(x)$ is 0" and write $\lim_{x \to \infty} H(x) = 0$.

Example. Page 172 number 34. Observe the "limits" of this function.

Definition. If as $x \to -\infty$ or as $x \to \infty$, the values of $R(x)$ approach some fixed number, L, then the line $y = L$ is a *horizontal asymptote* of the graph of R. If as x approaches some number c, the values $|R(x)| \to \infty$, then the line $x = c$ is a *vertical asymptote* of the graph of R. The graph of R never intersects a vertical asymptote.

Theorem. A rational function $R(x) = \dfrac{p(x)}{q(x)}$, in lowest terms, will have a vertical asymptote $x = r$ if r is a real zero of the denominator q. That is, if $x - r$ is a factor of the denominator q of the rational function R above, then R will have a vertical asymptote $x = r$.

Definition A rational function is *proper* if the degree of the numerator is less than the degree of the denominator. Otherwise, it is *improper*.

Theorem. If a rational function is proper, the line $y = 0$ is a horizontal asymptote of its graph.

Note. If a rational function $R(x) = \dfrac{p(x)}{q(x)}$ is improper, then we can use long division to write

$$R(x) = \frac{p(x)}{q(x)} = f(x) + \frac{r(x)}{q(x)}$$

where $f(x)$ is a polynomial and $\dfrac{r(x)}{q(x)}$ is a proper rational function. Since $\dfrac{r(x)}{q(x)}$ is proper, then $\dfrac{r(x)}{q(x)} \to 0$ as $x \to -\infty$ or as $x \to \infty$. As a result, $R(x) = \dfrac{p(x)}{q(x)} \to f(x)$ as $x \to -\infty$ or as $x \to \infty$. Depending on $f(x)$, we have three cases:

1. If $f(x) = b$, a constant, then the line $y = b$ is a horizontal asymptote of the graph of R.
2. If $f(x) = ax + b$, $a \neq 0$, then the line $y = ax + b$ is called an *oblique asymptote* (or *slant asymptote*) of the graph of R.
3. In all other cases, the graph of R approaches the graph of f, and there are no horizontal or oblique asymptotes.

Example. Page 172 number 50.