Chapter 2. Limits and Continuity

2.2. Limit of a Function and Limit Laws

Note. We have to be careful in our dealings with functions! Notice that $f(x) = \dfrac{x(x-1)}{x-1}$ and $g(x) = x$ are **NOT** the same functions! They do not even have the same domains. Therefore we cannot in general say $\dfrac{x(x-1)}{x-1} = x$. However, this equality holds if x lies in the domains of the functions. We *can* say:

$$\frac{x(x-1)}{x-1} = x \textbf{ IF } x \neq 1.$$

We can also say $f(x) = g(x)$ **IF** $x \neq 1$.

Definition. Informal Definition of Limit.

Let $f(x)$ be defined on an open interval about x_0, **except possibly at** x_0 **itself**. If $f(x)$ gets arbitrarily close to L for all x sufficiently close to x_0, we say that f *approaches the limit* L *as* x *approaches* x_0, and we write

$$\lim_{x \to x_0} f(x) = L.$$

Note. The above definition is **informal** (that is, it is not mathematically rigorous) since the terms "arbitrarily close" and "sufficiently close" are not defined.

Example. Evaluate $\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x(x-1)}{x-1}$.

Solution. From above, we see that

$$f(x) = \frac{x(x-1)}{x-1} = x \textbf{ IF } x \neq 1.$$

So $\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x(x-1)}{x-1} = \lim_{x\to 1} x$ if $x \neq 1$. Therefore, in words, we ask "what does x get close to when x is close to 1?" Well... x gets close to 1! So $\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x(x-1)}{x-1} = \lim_{x\to 1} x = 1$, even though we are not allowing x to equal 1; the important thing is that x can be made arbitrarily close to 1. Notice that this example shows that $\lim_{x\to 1} f(x) = \lim_{x\to 1} g(x)$, where $g(x) = x$. That is, the limits of f and g are the same, even though the functions f and g are different (though very subtly different—they only differ at $x = 1$).

Example. Example 1 page 65. Use the above technique to evaluate $\lim_{x\to 1} \frac{x^2-1}{x-1}$.

Note. Another very informal idea is the following:

Dr. Bob's Anthropomorphic Definition of Limit.

Let $f(x)$ be defined on an open interval about x_0, **except possibly at** x_0 **itself**. If the graph of $y = f(x)$ *tries to pass through the point* (x_0, L), then we say $\lim_{x \to x_0} f(x) = L$. Notice that it does not matter whether the graph actually passes through the point, only that it *tries to*.

Example. Page 73 number 2.

Theorem 1. Limit Rules.

If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule*: $\lim_{x \to c}(f(x) + g(x)) = L + M$.

2. *Difference Rule*: $\lim_{x \to c}(f(x) - g(x)) = L - M$.

3. *Constant Multiple Rule*: $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$.

4. *Product Rule*: $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$.

5. *Quotient Rule*: $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$.

6. *Power Rule*: If n is a positive integer, then $\lim_{x \to c}(f(x))^n = L^n$.

7. *Root Rule*: If n is a positive integer, then $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$ (if n is even, we also require that $\lim_{x \to c} f(x) = L > 0$).

Example. Page 74 number 52.

Theorem 2. Limits of Polynomials Can Be Found by Substitution.

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

Theorem 3. Limits of Rational Functions Can Be Found by Substituting IF the Limit of the Denominator Is Not Zero.

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Example. Page 74 numbers 14 and 18.

Theorem. Dr. Bob's Limit Theorem.

If $f(x) = g(x)$ for all x in an open interval containing c, except possibly c itself, then

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$$

provided these limits exist.

Note. Dr. Bob's Limit Theorem is a summary of what the text calls "Eliminating Zero Denominators Algebraically" and the use of "simpler fractions." The text also describes "Using Calculators and Computers to Estimate Limits" in which you plug in x-values "closer and closer" to x_0 to estimate $\lim_{x \to x_0} f(x)$. However, this instructor finds the use of such estimates horribly misleading! We will skip the exercises with instructions "make a table..."

Example. Page 74 numbers 34 and 38.

Theorem 4. Sandwich Theorem.

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

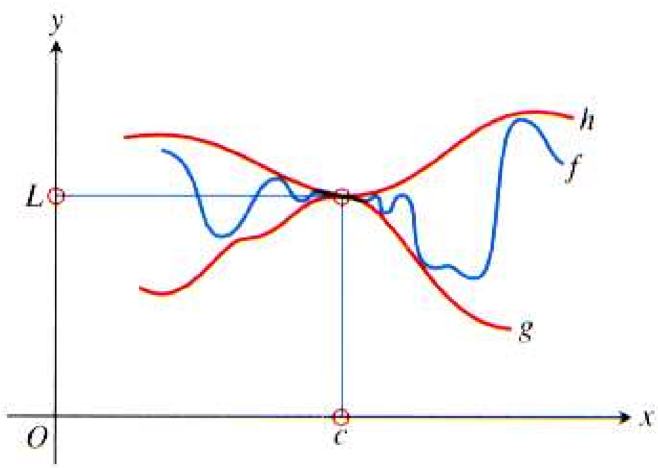

Figure 2.12, page 72

Example. Page 75 number 66a.

Example. Page 76 number 79.