Chapter 2. Limits and Continuity

2.5 Continuity

Definition. Continuity at a Point.

Interior Point: A function $y = f(x)$ is *continuous at an interior point* c of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is *continuous at a left endpoint* a or is *continuous at a right endpoint* b of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively.}$$

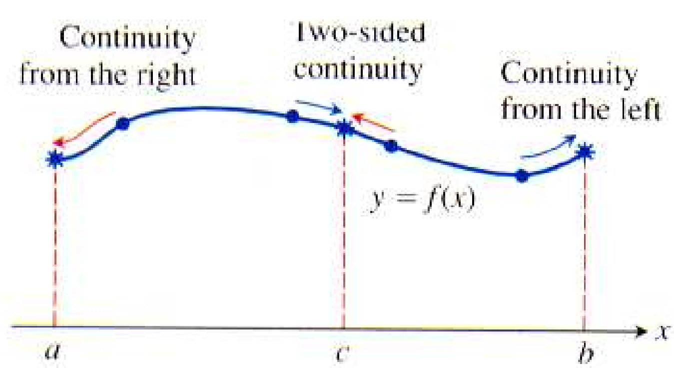

Figure 2.36, page 93.

Note. If a function is continuous at all interior points of its domain and the domain is an interval, then the function can be "drawn without picking up your pencil."

Example. Page 101 number 4.

Continuity Test.

A function $f(x)$ is continuous at an interior point of the domain of f, $x = c$, if and only if it meets the following three conditions:

1. $f(c)$ exists,

2. $\lim_{x \to c} f(x)$ exists, and

3. $\lim_{x \to c} f(x) = f(c)$.

Note. Polynomials, rational functions, and the six trigonometric functions are continuous at every point of their domains.

Example. Consider the piecewise defined function

$$f(x) = \begin{cases} x & \text{if } x \in (-\infty, 0) \\ 0 & \text{if } x = 0 \\ x^2 & \text{if } x \in (0, \infty) \end{cases}.$$

Is f continuous at $x = 0$?

Definition. A function f has a *removable discontinuity* at $x = a$ if $f(a)$ can be redefined in such a way that f is continuous at a. f has a *jump discontinuity* at $x = a$ if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist (as finite numbers) and are different. The book also gives examples of an *infinite discontinuity* and an *oscillating discontinuity* (see page 122).

Example. Discuss the discontinuities of $f(x) = \dfrac{|x|}{x}$ and $g(x) = \text{int } x$.

Theorem 8. Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums*: $f + g$

2. *Differences*: $f - g$

3. *Products*: $f \cdot g$

4. *Constant Multiples*: $k \cdot f$, for any number k

5. *Quotients*: f/g, provided $g(c) \neq 0$.

6. *Powers*: f^n, for a positive integer n

7. *Roots*: $\sqrt[n]{f}$, provided $\sqrt[n]{f}$ is defined on an open interval containing c, where n is a positive integer.

Theorem 9. Composite of Continuous Functions

If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

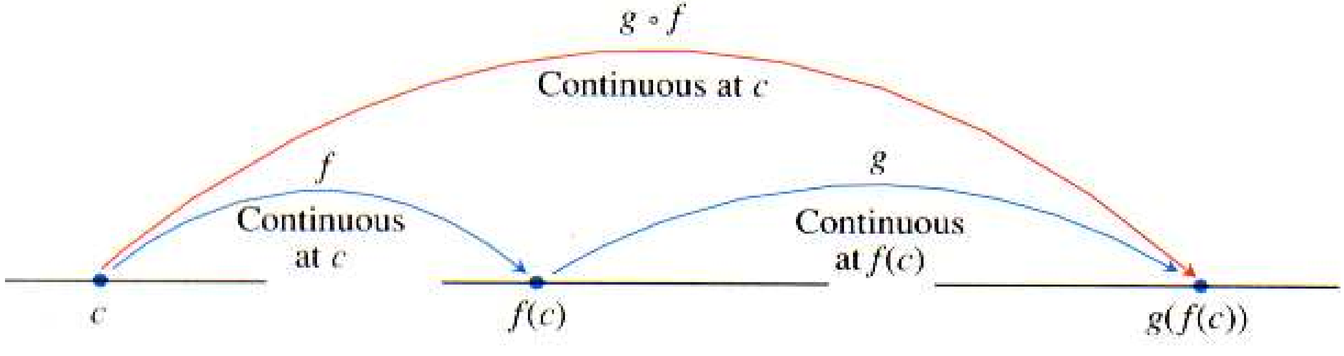

Figure 2.42, page 96.

Theorem 10. Limits of Continuous Functions.

If g is continuous at the point b and $\lim_{x \to c} f(x) = b$, the

$$\lim_{x \to c} g(f(x)) = g(b) = g(\lim_{x \to c} f(x)).$$

Example. Page 102 number 32.

Note. If a function has a removable discontinuity at a point, then we can redefine the function at that point in such a way as to create a new function which *is* continuous at that point. This new function is called a *continuous extension* of the original function.

Example. Page 102 number 40.

Theorem 11. The Intermediate Value Theorem for Continuous Functions

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

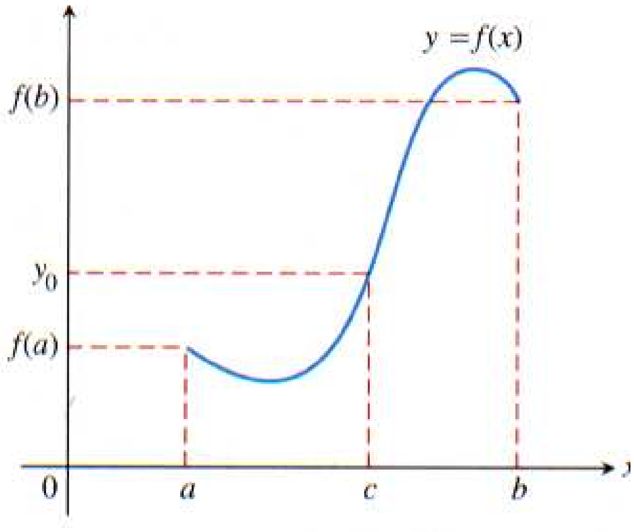

Figure from page 99.

Examples. Page 102 number 57a and Page 103 number 66.