Chapter 2. Limits and Continuity

2.6 Limits Involving Infinity; Asymptotes of Graphs

Definition. Formal Definition of Limits at Infinity.

1. We say that $f(x)$ has the *limit* L *as* x *approaches infinity* and we write

$$\lim_{x \to +\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

2. We say that $f(x)$ has the *limit* L *as* x *approaches negative infinity* and we write

$$\lim_{x \to -\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Definition. Informal Definition of Limits Involving Infinity.

1. We say that $f(x)$ has the *limit L as x approaches infinity* and write

$$\lim_{x\to+\infty} f(x) = L$$

if, as x moves increasingly far from the origin in the positive direction, $f(x)$ gets arbitrarily close to L.

2. We say that $f(x)$ has the *limit L as x approaches negative infinity* and write

$$\lim_{x\to-\infty} f(x) = L$$

if, as x moves increasingly far from the origin in the negative direction, $f(x)$ gets arbitrarily close to L.

Example. Example 1 page 104. Show that $\lim_{x\to\infty} \frac{1}{x} = 0$.

Solution. Let $\epsilon > 0$ be given. We must find a number M such that for all

$$x > M \quad \Rightarrow \quad \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \epsilon.$$

The implication will hold if $M = 1/\epsilon$ or any larger positive number (see the figure below). This proves $\lim_{x\to\infty} \frac{1}{x} = 0$. We can similarly prove that

$$\lim_{x\to-\infty}\frac{1}{x}=0.$$ *QED*

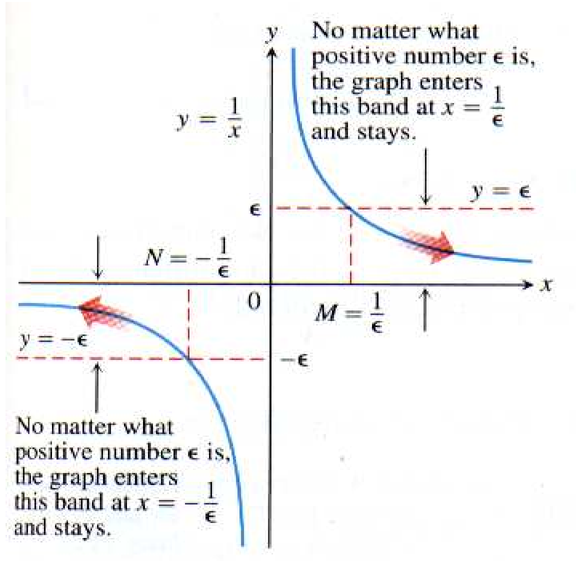


Figure 2.32, page 102 of the 11th Edition

Theorem 12. Rules for Limits as $x \to \pm\infty$.

If L, M, and k are real numbers and

$$\lim_{x\to\pm\infty} f(x) = L \quad \text{and} \quad \lim_{x\to\pm\infty} g(x) = M, \quad \text{then}$$

1. *Sum Rule*: $\lim_{x\to\pm\infty}(f(x)+g(x)) = L + M$

2. *Difference Rule*: $\lim_{x\to\pm\infty}(f(x)-g(x)) = L - M$

3. *Product Rule*: $\lim_{x\to\pm\infty}(f(x)\cdot g(x)) = L \cdot M$

4. *Constant Multiple Rule*: $\lim_{x\to\pm\infty}(k\cdot f(x)) = k \cdot L$

5. *Quotient Rule*: $\lim_{x\to\pm\infty} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, M \neq 0$

6. *Power Rule*: If n is a positive integer, then $\lim_{x\to\pm\infty} (f(x))^n = L^n$.

7. *Root Rule*: If n is a positive integer, then $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$ (if n is even, we also require that $\lim_{x\to c} f(x) = L > 0$).

Example. Page 114 number 14 and Page 115 number 36.

Definition. Horizontal Asymptote.

A line $y = b$ is a *horizontal asymptote* of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b.$$

Example. Page 115 number 68, find the horizontal asymptotes.

Example. Page 106 Example 5. Prove $\lim_{x\to-\infty} e^x = 0$.

Definition. Oblique Asymptotes.

If the degree of the numerator of a rational function is one greater than the degree of the denominator, the graph has an *oblique asymptote* (or *slant asymptote*). The asymptote is found by dividing the denominator into the numerator to express the function as a linear function plus a remainder that goes to zero as $x \to \pm\infty$.

Example. Page 116 number 102, find the slant asymptote.

Example. Page 116 number 86.

Definition. Infinity, Negative Infinity as Limits

1. We say that $f(x)$ *approaches infinity as* x *approaches* x_0, and we write

$$\lim_{x \to x_0} f(x) = \infty,$$

if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x

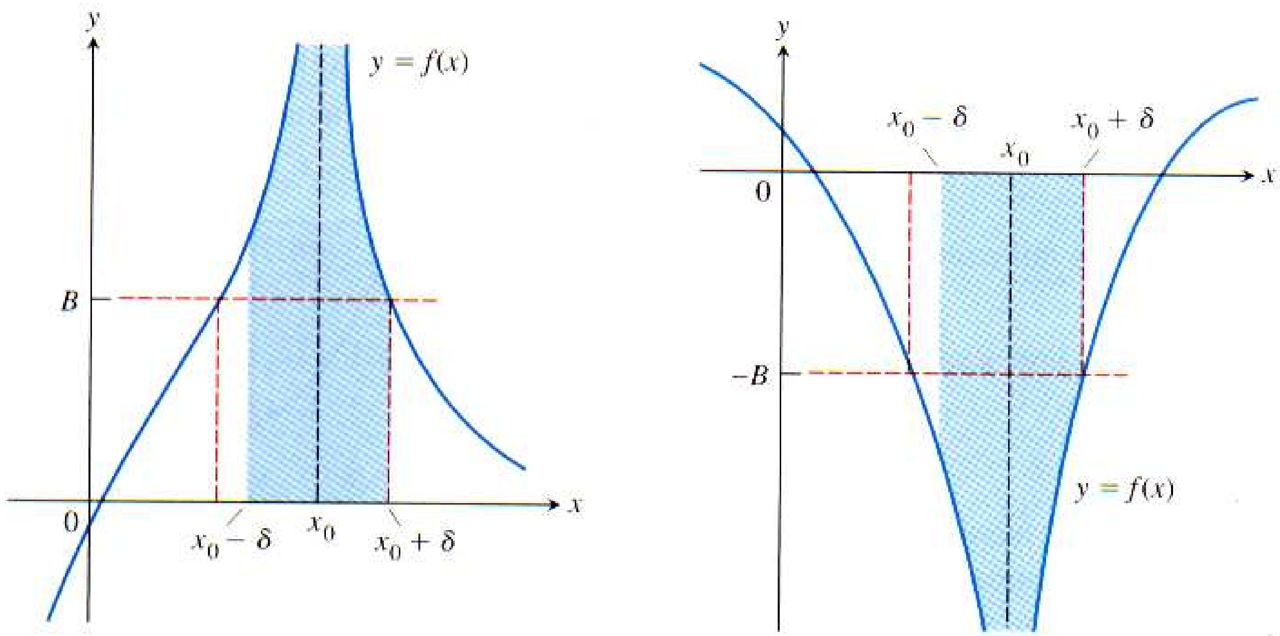
$$0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) > B.$$

2. We say that $f(x)$ *approaches negative infinity as* x *approaches* x_0, and we write

$$\lim_{x \to x_0} f(x) = -\infty,$$

if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) < -B.$$



Figures 2.40 and 2.41, 10th Edition.

Note. Informally, $\lim_{x \to x_0} f(x) = \infty$ if $f(x)$ can be made arbitrarily large by making x sufficiently close to x_0 (and similarly for f approaching negative infinity). We can also define one-sided infinite limits in an analogous manner (see page 116 number 93).

Definition. Vertical Asymptotes.

A line $x = a$ is a *vertical asymptote* of the graph if either

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty.$$

Note. Recall that we look for the vertical asymptotes of a rational function where the denominator is zero (though just because the denominator has zero at a point, the function does not *necessarily* have a vertical asymptote at that point). We make things more precise in the following result:

Dr. Bob's Infinite Limits Theorem. Let $f(x) = \dfrac{p(x)}{q(x)}$. Suppose $\lim_{x\to x_0} p(x) = L \neq 0$, $\lim_{x\to x_0} q(x) = 0$, and $q(x)$ is of the same sign in some open interval containing x_0. Then $\lim_{x\to x_0} f(x) = \pm\infty$. We can say something similar for one-sided limits.

Note. We can simplify Dr. Bob's Infinite Limits Theorem by applying it to rational functions. It then becomes: "Let $f(x) = \dfrac{p(x)}{q(x)}$ be a rational function. Suppose $\lim_{x\to x_0^+} p(x) = L \neq 0$ and $\lim_{x\to x_0^+} q(x) = 0$. Then $\lim_{x\to x_0^+} f(x) = \pm\infty$." We can say something similar for limits from the left and for two-sided limits.

Examples. Page 115 numbers 54, Page 116 number 102 (again), Page 115 number 74, Page 116 number 96, and Page 112 Example 18.