Chapter 4. Applications of Derivatives

4.5 Indeterminate Forms and L'Hôpital's Rule

Definition. We say that $\lim_{x \to a} \dfrac{f(x)}{g(x)}$ is in

1. $0/0$ *indeterminate form* if

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0,$$

2. ∞/∞ *indeterminate form* if

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty,$$

The following theorem allows us to deal with the $0/0$ indeterminate form.

Theorem 6. L'Hôpital's Rule. Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)},$$

assuming that the limit on the right side of this equation exists.

Proof. See page 260. *QED*

Example. Use L'Hôpital's Rule to show $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim_{x\to 0} \frac{1-\cos x}{x}$.

Example. Page 261 number 16.

Note. L'Hôpital's Rule also applies to ∞/∞ indeterminate forms. In fact, we can often convert $\infty - \infty$ and $0 \times \infty$ forms into $0/0$ or ∞/∞ forms. The rule also applies to one-sided limits which satisfy the appropriate hypotheses.

Example. Page 261 number 38 and 46, page 262 number 81b.

Theorem. If $\lim_{x\to a} \ln f(x) = L$ then

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^L.$$

Here, a may be finite or infinite.

Note. The proof of the previous theorem follows from the continuity of the exponential function at every real number. This result allows us to extend L'Hopital's Rule to indeterminate forms 1^∞, 0^0, and ∞^0.

Example. Page 261 numbers 52 and 58.