Chapter 4. Applications of Derivatives

4.8 Antiderivatives

Definition. A function $F(x)$ is an *antiderivative* of a function $f(x)$ if $F'(x) = f(x)$ for all x in the domain of f. The most general antiderivative (which is really the **set** of all antiderivatives) of f is the *indefinite integral* of f with respect to x, denoted by $\int f(x)\,dx$. The symbol $\int$ is an *integral sign.* The function f is the *integrand* of the integral, and x is the *variable of integration.*

Note. We denote the indefinite integral (set) as

$$\int f(x)\,dx = F(x) + C$$

where F is a specific antiderivative and C represents an "arbitrary constant." (In class, we will use "k" for a specific constant.)

Note. In terms of the notation of indefinite integrals, we have (from Table 4.2; with $k = 1$ we get the table after the following one):

Indefinite Integral	**Derivative Formula**
1. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, n \neq -1,$	$\frac{d}{dx}\left[\frac{x^{n+1}}{n+1}\right] = x^n$
2. $\int \sin kx \, dx = -\frac{\cos kx}{k} + C$	$\frac{d}{dx}\left[-\frac{\cos kx}{k}\right] = \sin kx$
3. $\int \cos kx \, dx = \frac{\sin kx}{k} + C$	$\frac{d}{dx}\left[\frac{\sin kx}{k}\right] = \cos kx$
4. $\int \sec^2 kx \, dx = \frac{1}{k}\tan kx + C$	$\frac{d}{dx}[\tan kx] = k\sec^2 kx$
5. $\int \csc^2 kx \, dx = -\frac{1}{k}\cot kx + C$	$\frac{d}{dx}[-\cot kx] = k\csc^2 kx$
6. $\int \sec kx \tan kx \, dx = \frac{1}{k}\sec kx + C$	$\frac{d}{dx}[\sec kx] = k\sec kx \tan kx$
7. $\int \csc kx \cot kx \, dx = -\frac{1}{k}\csc kx + C$	$\frac{d}{dx}[-\csc kx] = k\csc kx \cot kx$
8. $\int e^{kx} \, dx = \frac{1}{k}e^{kx} + C$	$\frac{d}{dx}\left[\frac{1}{k}e^{kx}\right] = e^{kx}$
9. $\int \frac{1}{x} \, dx = \ln\lvert x\rvert + C,\ x \neq 0$	$\frac{d}{dx}[\ln x] = \frac{1}{x}$
10. $\int \frac{1}{\sqrt{1-k^2x^2}} \, dx = \frac{1}{k}\sin^{-1} kx + C$	$\frac{d}{dx}\left[\frac{1}{k}\sin^{-1} kx\right] = \frac{1}{\sqrt{1-k^2x^2}}$
11. $\int \frac{1}{1+k^2x^2} \, dx = \frac{1}{k}\tan^{-1} kx + C$	$\frac{d}{dx}\left[\frac{1}{k}\tan^{-1} kx\right] = \frac{1}{1+k^2x^2}$
12. $\int \frac{1}{x\sqrt{k^2x^2-1}} \, dx = \sec^{-1} kx + C,\ kx > 1$	$\frac{d}{dx}[\sec^{-1} kx] = \frac{1}{x\sqrt{k^2x^2-1}}$
13. $\int a^{kx} \, dx = \left(\frac{1}{k\ln a}\right)a^{kx} + C,\ a > 0,\ a \neq 1$	$\frac{d}{dx}[a^{kx}] = a^{kx}\ln a$

Indefinite Integral	**Derivative Formula**
1. $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + C, n \neq -1,$	$\dfrac{d}{dx}\left[\dfrac{x^{n+1}}{n+1}\right] = x^n$
2. $\int \sin x \, dx = -\cos x + C$	$\dfrac{d}{dx}[-\cos x] = \sin x$
3. $\int \cos x \, dx = \sin x + C$	$\dfrac{d}{dx}[\sin x] = \cos x$
4. $\int \sec^2 x \, dx = \tan x + C$	$\dfrac{d}{dx}[\tan x] = \sec^2 x$
5. $\int \csc^2 x \, dx = -\cot x + C$	$\dfrac{d}{dx}[-\cot x] = k \csc^2 x$
6. $\int \sec x \tan x \, dx = \sec x + C$	$\dfrac{d}{dx}[\sec x] = \sec x \tan x$
7. $\int \csc x \cot x \, dx = -\csc x + C$	$\dfrac{d}{dx}[-\csc x] = \csc x \cot x$
8. $\int e^x \, dx = e^x + C$	$\dfrac{d}{dx}[e^x] = e^x$
9. $\int \dfrac{1}{x} \, dx = \ln\lvert x\rvert + C, \; x \neq 0$	$\dfrac{d}{dx}[\ln x] = \dfrac{1}{x}$
10. $\int \dfrac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$	$\dfrac{d}{dx}\left[\sin^{-1} x\right] = \dfrac{1}{\sqrt{1-x^2}}$
11. $\int \dfrac{1}{1+x^2} \, dx = \tan^{-1} x + C$	$\dfrac{d}{dx}\left[\tan^{-1} x\right] = \dfrac{1}{1+x^2}$
12. $\int \dfrac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1} x + C, \; x > 1$	$\dfrac{d}{dx}[\sec^{-1} x] = \dfrac{1}{x\sqrt{x^2-1}}$
13. $\int a^x \, dx = \left(\dfrac{1}{\ln a}\right) a^x + C, \; a > 0, \; a \neq 1$	$\dfrac{d}{dx}[a^x] = a^x \ln a$

Note. Based on the properties of differentiation, we have the following "linearity rules" for indefinite integrals. Suppose F is an antiderivative of f, G is an antiderivative of g, and k is a constant.

1. Constant Multiple rule: $\int kf(x)\,dx = k\int f(x)\,dx = kF(x) + C$.

2. Negative Rule: $-f(x)\,dx = -\int f(x)\,dx = -F(x) + C$.

3. Sum or Difference Rule: $\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int G(x)\,dx = F(x) \pm G(x) + C$.

Examples. Page 285 number 32 and Page 286 numbers 54 and 66.

Definition. A *differential equation* is an equation relating an unknown function y of x and one or more of its derivatives. A function whose derivatives satisfy a differential equation is called a *solution* of the differential equation and the set of all solutions is called the *general solution.* The problem of finding a specific function y of x which is a solution to a differential equation and satisfies certain *initial condition(s)* of the form $y(x_0) = y_0$, $y'(x_0) = y'_0$, etc., is called an *initial value problem.*

Examples. Page 287 number 116 and 102, Page 288 number 120.