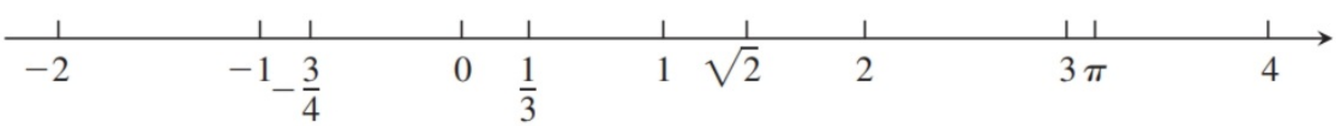
Appendices

A.1. Real Numbers and the Real Line

Note. In this appendix we discuss the real numbers and other collections of numbers. We represent the real numbers "geometrically" with the real number line. We consider sets of real numbers, intervals, inequalities, absolute values, and the interactions of these ideas.

Note. We deal with the real numbers informally in this appendix. See Appendix A.6. Theory of the Real Numbers for a somewhat more formal approach. The real numbers are developed axiomatically as a "complete ordered field" in ETSU's Analysis 1 (MATH 4217/5227); see my online notes for Section 1.2. Properties of the Real Numbers as an Ordered Field and Section 1.3. The Completeness Axiom.

Note/definition. *Thomas' Calculus* states that the "*real numbers* are the numbers that can be expressed as decimals. . . ." The real numbers can then be associated with the points on the *real number line*:



This is often called the *geometric representation* of the real numbers. The real numbers are denoted $\mathbb{R}$.

Note/definition. The *algebraic properties* (or *field properties*) of the real numbers deal with how the numbers interact under the operations of addition and multiplication. (The text book includes the operations of subtraction and division in these properties, but neither is actual a mathematical operation! Subtraction is just shorthand notation for adding an additive inverse and division is shorthand for multiplying be a multiplicative inverse.) The additive identity zero, denoted "0," does not have a multiplicative inverse (and so there is no division by 0).

Note/definition. The *order properties* of the real numbers deal with the properties of the inequalities "greater than," $>$, and "less than," $<$. For distinct real numbers a and b we write $a < b$ or $b > a$ when a is to the left of b on the real number line (or, equivalently, b is the to right of a on the real number line). A list of properties of inequalities which we assume is:

Let a, b, and c be real numbers.

1. If $a < b$ then $a + c < b + c$.

2. If $a < b$ then $a - c < b - c$.

3. If $a < b$ and $c > 0$ then $ac < bc$.

4. If $a < b$ and $c < 0$ then $bc < ac$.

5. If $a > 0$ then $1/a > 0$.

6. For a and b both positive or both negative, if $a < b$ then $1/b < 1/a$.

Note/definition. The *completeness property* of the real numbers deals with the fact that the real numbers form a continuum. As a consequence, the real number line has no "holes" (or "gaps") in it. This property is essential when considering limits, which we cover in Chapter 2.

Note/definition. We often use set notation and terminology. A *set* is a collection of objects (usually real numbers for us), called *elements* of the set. For set S, we write $a \in S$ to indicate that a is an element of set S. We write $a \notin S$ if a is not an element of set S. For sets S and T, the *union* of these two sets, $S \cup T$, is the set consisting of all elements in either set S or in set T. For sets S and T, the *intersection* of these two sets, $S \cap T$, is the set consisting of all elements in both sets S and T. The *empty set* is the set with no elements, denoted $\varnothing$. Set S is a subset of set T, denoted $S \subseteq T$, if every element of S is also an element of T; T is then called a *superset* of S.

Note. We sometimes present a set by listing its elements within a pair of set brackets: $A = \{1, 2, 3, 4, 5\}$. We often describe subsets of a given set by putting a condition on the elements of the subset: $A = \{x \in \mathbb{Z} \mid 0 < x < 6\}$. This is read "Set A equals the set of all x in the integers such that 0 is less than x is less than 6." Notice that this is the set $\{1, 2, 3, 4, 5\}$ again.

Note/definition. We are interested in certain special subsets of the real numbers.

1. The *natural numbers*, $\mathbb{N}$, is the set of numbers $\{1, 2, 3, 4, \ldots\}$.

2. The *integers*, $\mathbb{Z}$, is the set of numbers $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

3. The *rational numbers*, $\mathbb{Q}$, is the set of numbers $\{m/n \mid m, n \in \mathbb{Z} \text{ and } n \neq 0\}$.

4. The *irrational numbers* is the set of real numbers which are not rational: $\{x \in \mathbb{R} \mid x \notin \mathbb{Q}\}$.

Note. We are not too concerned about the sizes of sets, but a funny story is that $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are all infinite sets. However, the sets $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are all sets of the same size (they are "countable") and the sets $\mathbb{R}$ and the set of irrational numbers are the same size (they are "uncountable"). The surprise is that the countable sets are not the same size as the uncountable sets! This means that some infinite sets are larger than others!!! For more details, see my online notes for Analysis 1 (MATH 4217/5217) on 1.3. The Completeness Axiom; in particular, see the definition of "same cardinality" sets, Theorem 1-20, and Theorem 1-21 (Cantor's Theorem).

Definition. A subset of the real numbers is an *interval* if it contains at least two numbers and contains all the real numbers lying between any two of its elements.

Note. We often describe intervals using inequalities or "interval notation." Table A.1 contains an example of each type of interval, along with the notation used to represent it.

TABLE A.1 Types of intervals

Notation	Set description	Type	Picture
(a, b)	$\{x \mid a < x < b\}$	Open	
$[a, b]$	$\{x \mid a \leq x \leq b\}$	Closed	
$[a, b)$	$\{x \mid a \leq x < b\}$	Half-open	
$(a, b]$	$\{x \mid a < x \leq b\}$	Half-open	
(a, ∞)	$\{x \mid x > a\}$	Open	
$[a, \infty)$	$\{x \mid x \geq a\}$	Closed	
$(-\infty, b)$	$\{x \mid x < b\}$	Open	
$(-\infty, b]$	$\{x \mid x \leq b\}$	Closed	
$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

Definition. Notice that some intervals have one endpoint and some have two endpoints (and one interval, $(-\infty, \infty) = \mathbb{R}$, has no endpoints). An interval is *closed* if it contains its endpoint(s). An interval is *open* if it does not contain its endpoint(s). An interval is *half-open* if it has two endpoints and it contains exactly one of them. An interval with two endpoints is a *finite interval*. If an interval is not a finite interval then it is an *infinite interval*. The endpoints of an interval are called the *boundary points* of the interval, and non-boundary points in an interval are *interior points* of the interval.

Note. We often express solutions to inequalities in terms of intervals.

Example. Exercise A.1.6. Find all $x \in \mathbb{R}$ satisfying $\frac{4}{5}(x-2) < \frac{1}{3}(x-6)$ and show the solution set on the real number line.

Definition. The *absolute value* function of $x \in \mathbb{R}$, is defined as

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

Note. We use the absolute value function to measure the distance between two points on the real number line. If $a, b \in \mathbb{R}$, then the distance between a and b is $|a-b| = |b-a|$. When addressing limits in Chapter 2, we will have definitions based on this measure of distance.

Note. Notice that we can algebraically define $|x|$ as $|x| = \sqrt{x^2}$. Remember, for any nonnegative real number x, $\sqrt{x} \geq 0$. That is, square roots (when they exist) are never negative! So if we write $\sqrt{9}$ then we mean 3. **We do not have** $\sqrt{9} = \pm 3$**!!!** The square root symbol represents the non-negative square root; if you want both the positive and negative square roots of 9 then you need to "ask" for them: $\pm\sqrt{9} = \pm 3$.

Note. A list of properties of absolute value is the following:

Let a and b be real numbers.

1. $|-a| = |a|$.

2. $|ab| = |a|\,|b|$.

3. $|a/b| = |a|/|b|$ where $b \neq 0$.

4. $|a+b| \leq |a| + |b|$.

Property 4 is called the Triangle Inequality.

Example. Exercise A.1.24. Prove the Triangle Inequality.

Note. We can relate intervals to absolute values as follows:

Let a be a positive real number.

5. $|x| = a$ if and only if $x = \pm a$.

6. $|x| < a$ if and only if $-a < x < a$.

7. $|x| > a$ if and only if $x < -a$ or $x > a$.

8. $|x| \leq a$ if and only if $-a \leq x \leq a$.

9. $|x| \geq a$ if and only if $x \leq -a$ of $x \geq a$.

Example. Exercise A.1.12, Exercise A.1.16, and Exercise A.1.20.

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