Appendices

A.2. Mathematical Induction

Note. In this appendix we state the mathematical induction principle and use it to give proofs of some results used previously. In particular, we give proofs of the two summation theorems of Section 5.2. Sigma Notation and Limits of Finite Sums. Namely, we prove Theorem 5.2.A, "Algebra for Finite Sums," and Theorem 5.2.B, "The Sum of Powers of the First n Natural Numbers."

Note. "The Induction Principle" is a result from set theory. For a discussion, see my online notes for Discrete Math (MATH 2710; this class is no longer in the ETSU repertoire) on 3.2. Mathematical Induction; this topic would be covered in Mathematical Reasoning (MATH 3000) as well. This result follows from the very definition of the natural numbers $\mathbb{N}$. It states:

> **The Induction Principle.** Let $\mathbf{P}(x)$ be a property (possibly with parameters). Assume that
>
> (a) $\mathbf{P}(0)$ holds.
>
> (b) For all $n \in \mathbb{N}$, $\mathbf{P}(n)$ implies $\mathbf{P}(n+1)$.
>
> Then $\mathbf{P}$ holds for all natural numbers $n \in \mathbb{N}$.

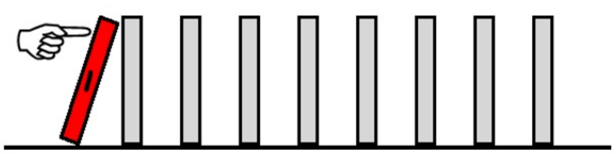
See my online notes for Set Theory (not a formal ETSU class), in particular see "Section 3.2. Properties of Natural Numbers."

Note. *Thomas' Calculus* calls The Induction Principle "the *mathematical* induction principle." In connection with establishing a formula which is a function of natural number n (such as the sum of the first n natural numbers), our text states that the technique of proof by induction requires two steps:

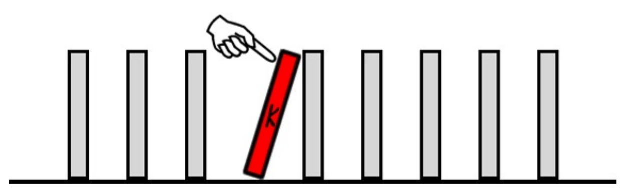
- Check that the formula holds for $n = 1$.
- Prove that *if* the formula holds for any natural number $n = k$, *then* it also holds for the next natural number, $n = k + 1$.

This is the same as the statement of The Induction Principle above, except that Thomas starts at $n = 1$ and the Induction Principle starts at $n = 0$. The "property **P**" of The Induction Principle corresponds to Thomas' "formula."

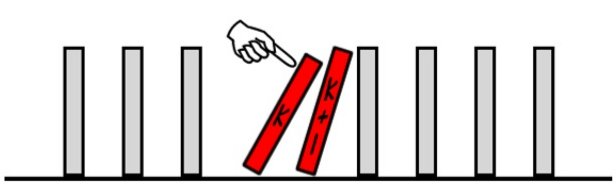
Note. Think of mathematical induction in terms of falling dominos. The main idea is that to knock down a linear arrangement of dominos, you need to knock down the first domino and make sure that when a domino falls then it knocks down the next domino. A nice collection of illustrations of this idea is the following, from coolmath.com:



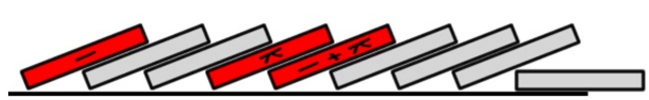
Check that the formula holds for $n = 1$.



Assume the formula holds for $n = k$.



Show that the assumption *implies* that the formula holds for $n = k + 1$.



Conclude that the formula holds for all $n \in \mathbb{N}$.

Example. We now illustrate the use of mathematical induction by establishing one of the formulae used in 5.2. Sigma Notation and Limits of Finite Sums (this is Theorem 5.2.B(2)):

Example A.2.1. Use mathematical induction to prove that for natural number $n \in \mathbb{N}$,

$$1 + 2 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Note. Thomas comments that we need not start at natural number 1 in using mathematical induction, but instead could start at any integer $n_1 \in \mathbb{Z}$ and then use induction to prove the validity of a formula of an integer n for which $n \geq n_1$.

Example A.2.A. Prove that for differentiable functions of x, $u_1, u_2, \ldots, u_n$, we have

$$\frac{d}{dx}[u_1 + u_2 + \cdots + u_n] = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx}.$$

This is a generalization of the Derivative Sum Rule (Theorem 3.3.E). We used this result several times in Chapters 3 and 4 without ever explicitly stating so.

Example. Exercise A.2.2. This result will be useful when we address geometric series in Calculus 2 (MATH 1920).

Example. Exercise A.2.9, "Sums of Squares." Prove Theorem 5.2.B(2):

$$1^2 + 2^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \text{ for all } n \in \mathbb{N}.$$

Example. Exercise A.2.10, "Sums of Cubes." Prove Theorem 5.2.B(3):

$$1^3 + 2^3 + \cdots + n^3 = \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 \text{ for all } n \in \mathbb{N}.$$

Notice that Example A.2.1, Exercise A.2.9, and Exercise A.2.10 completes the proof of Theorem 5.2.B, "The Sum of Powers of the First n Natural Numbers.

Example. Exercise A.2.11. Prove Theorem 5.2.A, "Algebra for Finite Sums."

Example A.2.B. Prove the General Product Rule (see Exercise 3.3.77 for motivation of this result): For differentiable functions $u_1, u_2, \ldots, u_n$, we have that the derivative of the product $u_1 u_2 \cdots u_n$ exists and

$$\begin{aligned}
\frac{d}{dx}[(u_1)(u_2)\cdots(u_n)] \quad = \quad & [u_1'](u_2)(u_3)\cdots(u_{n-1})(u_n) \\
& +(u_1)[u_2'](u_3)\cdots(u_{n-1})(u_n) \\
& +(u_1)(u_2)[u_3']\cdots(u_{n-1})(u_n) + \cdots \\
& +(u_1)(u_2)(u_3)\cdots[u_{n-1}'](u_n) \\
& +(u_1)(u_2)(u_3)\cdots(u_{n-1})[u_n'].
\end{aligned}$$

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