Appendices

A.4. Proofs of Limit Theorems

Note. In this appendix we prove some parts of Theorem 2.1, Limit Rules. We give \( \varepsilon/\delta \) proofs, based on the definition of limit, of the Product Rule for Limits and the Quotient Rule for Limits of Theorem 2.1, plus a proof of the Sandwich Theorem, Theorem 2.4 (all from Section 2.2).

**Theorem 2.1(4). Limit Product Rule.**

If \( \lim_{x \to c} f(x) = L \) and \( \lim_{x \to c} g(x) = M \), then

\[
\lim_{x \to c} (f(x)g(x)) = \left( \lim_{x \to c} f(x) \right) \left( \lim_{x \to c} g(x) \right) = LM.
\]

**Theorem 2.1(5). Limit Quotient Rule.**

If \( \lim_{x \to c} f(x) = L \) and \( \lim_{x \to c} g(x) = M \), then

\[
\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{L}{M},
\]

if \( \lim_{x \to c} g(x) = M \neq 0 \).

**Theorem 2.4. Sandwich Theorem.**

Suppose that \( g(x) \leq f(x) \leq h(x) \) for all \( x \) in some open interval containing \( c \), except possibly at \( x = c \) itself. Suppose also that

\[
\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.
\]

Then \( \lim_{x \to c} f(x) = L \).

Revised: 8/9/2020