Appendices

A.6. Theory of the Real Numbers

Note. In this appendix we attempt to rigorously define the real numbers. This information is normally covered in ETSU's Analysis 1 (MATH 4217/5227); see my online notes for Section 1.2. Properties of the Real Numbers as an Ordered Field and Section 1.3. The Completeness Axiom. Though a foundational part of calculus, this material is not essential for the understanding of Calculus 1.

Note. We state the definition of the real numbers in terms of axioms. We have algebraic axioms (or the "field axioms" which give the existence of certain real numbers and describe how addition and multiplication interact), order axioms (which describe the ideas of "greater than" and "less than"), and the completeness property (which makes the real numbers a continuum).

Definition. The real numbers $\mathbb{R}$ satisfy the following algebraic properties (these are the *field axioms*):

A1. $a + (b + c) = (a + b) + c$ for all $a, b, c \in \mathbb{R}$, Associativity of Addition.

A2. $a + b = b + a$ for all $a, b \in \mathbb{R}$, Commutitivity of Addition.

A3. There exists a real number denoted "0" such that $a + 0 = a$ for all $a \in \mathbb{R}$, Additive Identity.

A4. For each $a \in \mathbb{R}$, there is a number b such that $a + b = 0$, Additive Inverses.

M1. $a(bc) = (ab)c$ for all $a, b, c \in \mathbb{R}$, Associtivity of Multiplication.

M2. $ab = ba$ for all $a, b \in \mathbb{R}$, Commutivity of Multiplication.

M3. There exists a real number denoted "1" such that $a1 = a$ for all $a \in \mathbb{R}$, Multiplicative Identity.

M4. For each nonzero $a \in \mathbb{R}$ there is a number $b \in \mathbb{R}$ such that $ab = 1$, Multiplicative Inverses.

D. $a(b+c) = ab+ac$ for all $a, b, c \in \mathbb{R}$, Distribution of Multiplication over Addition.

Note. Any algebraic structure satisfying the field axioms is called a "field." In addition to the real numbers $\mathbb{R}$, the rational numbers $\mathbb{Q}$ and the complex numbers $\mathbb{C}$ are fields.

Definition. The real numbers $\mathbb{R}$ satisfy the following order properties (these are the *order axioms*):

O1. For any $a, b \in \mathbb{R}$, either $a \leq b$ or $b \leq a$, Comparability.

O2. If $a \leq b$ and $b \leq a$ then $a = b$, Law of Trichotomy.

O3. If $a \leq b$ and $b \leq c$ then $a \leq c$, Transitivity.

O4. If $a \leq b$ then $a + c \leq b + c$, Preservation of $\leq$ Under Addition.

O5. If $a \leq b$ and $0 \leq c$ then $ac \leq bc$, Preservation of $\leq$ Under Multiplication by Non-negative numbers.

Note. A field that satisfies the order axioms is an "ordered field." In addition to the real numbers $\mathbb{R}$, the rational numbers $\mathbb{Q}$ are an ordered field. The complex numbers $\mathbb{C}$ are a field but are not ordered (see my online notes on Ordering the Complex Numbers).

Note. For the final axiom of the real numbers, we need a few additional definitions.

Definition. A number M is an *upper bound* for a set of numbers if all numbers in the set are less than or equal to M. M is a *least upper bound* for a set S of numbers if it is an upper bound for set S and no number $N < M$ is also an upper bound of S. An ordered field is *complete* if every nonempty set of field elements which is bounded above has a least upper bound.

Definition. The real numbers satisfy the completeness property:

C1. The real numbers are complete, Axiom of Completeness.

Note. We therefore have that the real numbers are a complete ordered field. Notice that the rational numbers $\mathbb{Q}$ are not complete since the set $S = \{x \in \mathbb{Q} \mid x^2 < 2\}$ of rational numbers has a rational upper bound (say 3), but it does not have a rational least upper bound! Any candidate rational least upper bound would have to be bigger than the irrational number $\sqrt{2}$, so there is another rational number between $\sqrt{2}$ and the candidate rational least upper bound and this other number is also an upper bound of S, contradicting the possibility that the candidate is a *least* upper bound of S.

Note. So the real numbers are *a* complete ordered field. . . but are there other complete ordered fields? In fact, we can say that the real numbers are *the* complete ordered fields, since it can be shown that all complete ordered fields are the same (they are "isomorphic"). Details on this can be found in *Which Numbers are Real?* by Michael Henle, Washington, DC: Mathematical Association of America, Inc. (2012) (see Theorem 2.3.3 of page 48).

Note. *Thomas' Calculus* states in Appendix A.6 that "The completeness property is at the heart of many results in calculus." Without completeness, the existence of limits would not be insured. You have an intuitive feel for what a *continuum* is, and it is really the Axiom of Completeness that makes the real numbers a continuum (that is, the real number line has no holes or gaps in it).

Note. In connection with the idea of completeness and of a continuum, consider the following story. Imagine that an airplane taxis down a runway (at height 0) and takes to the air. Once in the air (i.e., when the height is positive), the plane remains in the air (instead of, say, the wheels bouncing on the runway before the plane gains altitude). Also, the plane is either in the air (with positive height) or on the ground (as long as a wheel is on the ground, we say that the plane is on the ground and that the height is 0; the plane is not in the air until all parts of the plane are off the ground). We ask the question: "Is there

(a) a first point (in time) at which the plane is in the air,

(b) a last point (in time) at which the plane is on the ground,

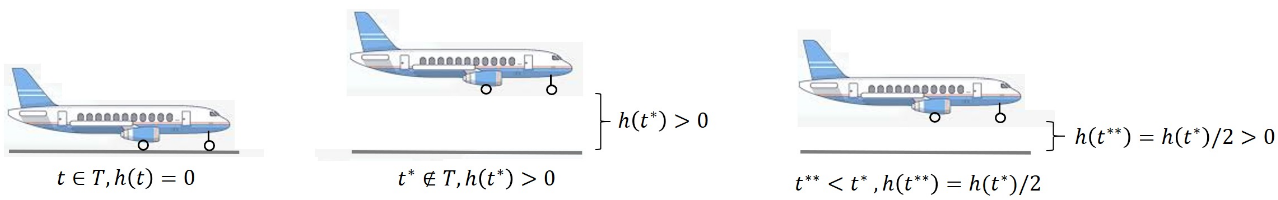
(c) both (a) and (b), or

(d) neither (a) nor (b)?"

My experience has revealed that students like answer (c). I fear this is because they are thinking of the numbers as distributed one-after-another along the real number line. But this isn't the case! Between any two real numbers there is another real number. So if there is both a last point in time at which the plane was on the ground and a first point in time at which the plane was in the air, then where is it between these two times? It can't be on the ground since these are times after the last point in time that it was on the ground. It can't be in the air since these are times before the first point in time that it is in the air! So the answer cannot be (c).

Let $h(t)$ be the function that gives the height of the plane at time t (so when $h(t) = 0$ the plane is "on the ground" and when $h(t) > 0$ then the plane is "in the air"). Consider the set of times that the plane is on the ground with height $h(t) = 0$, say $T = \{t \in \mathbb{R} \mid h(t) = 0\}$. Then T is a set of real numbers and, since the plane does eventually take off, then set T has an upper bound. By the Axiom of Completeness, set T has a least upper bound, say t_ℓ. Since t_ℓ is the least upper bound for T, for all times $t < t_\ell$ the plane must be on the ground with $h(t) = 0$, and for all times $t > t_\ell$ the plane must be in the air with $h(t) > 0$. So the question becomes $h(t_\ell) =?$

Suppose $h(t^*) > 0$ for some time t^*. At some earlier time, say t^{**}, we must have had the height of the plane as $h(t^*)/2$ which is also positive (see the figure below). So there cannot be a first point in time when the plane is in the air! Implicit in this story is that the function h is continuous (so we do not allow quantum leaps in height!); we are actually applying the Intermediate Value Theorem (Theorem 2.11) here. So we cannot have $h(t_\ell) > 0$ (since $h(t) = 0$ for all $t < t_\ell$) and it must be that $h(t_\ell) = 0$. That is, there is a last point in time that the plane is on the ground and the answer is (b).



Notice that the existence of t_ℓ is given by the Axiom of Completeness. If we only considered rational times, then the above argument falls apart and it could be that the answer is (d) (if, say, $h(t) = 0$ for $t \in \mathbb{Q}$ and $t < \sqrt{2}$, and $h(t) > 0$ for $t \in \mathbb{Q}$ and $t > \sqrt{2}$).

Note. The Axiom of Completeness was first stated by Richard Dedekind (October 6, 1831 - February 12, 1916) in his 1872 work "Continuity and Irrational Numbers" (a copy can be found online at Project Gutenberg). His approach did not use the language of "upper bounds" and "least upper bounds," but instead "cuts" of the real number line (what today is called a "Dedekind cut"). It is surprising that a rigorous definition of the real numbers only dates from 150, or so, years ago!

Image from MacTutor History of Mathematics Archive

A *Dedekind cut* is a partitioning of the number line into two sets A and B such that every element of set A is less than every element of set B (symbolically, $a \in A$ and $b \in B$ implies $a < b$, $A \cap B = \varnothing$, and $A \cup B = \mathbb{R}$). The Axiom of Completeness then is stated as: "Exactly one of the following holds: (1) there is a largest number in set A, or (2) there is a least number in set B." In terms of the airplane example, we have that the values of t for which the plane is on the ground form set A (that is, $A = T$), and the values of t for which the plane is in the air form set B. In that example, there is a largest number in set A.

Revised: 8/23/2020