Chapter 1. Functions

Note. We review properties of functions and their graphs. In particular, we review trigonometric (in Section 1.3), exponential (in Section 1.5), and logarithmic (in Section 1.6) functions. This is a quick recap of the material of Precalculus 1 (algebra) and Precalculus 2 (Trigonometry).

1.1. Functions and Their Graphs

Note. In this section we review several properties of functions and present a number of graphs of functions. We start by assuming that you are familiar with the idea of a "set" and the set theoretic symbol "$\in$" ("an element of"). For more details, see Appendix 1. Real Numbers and the Real Line

Definition. A *function* f from a set D to a set Y is a rule that assigns a unique element $f(x) \in Y$ to each element $x \in D$. The symbol f represents the function, the letter x is the *independent variable* representing the input value of f, and y is the *dependent variable* or output value of f at x. The set D of all possible input values is called the *domain* of the function. The set of all values of $f(x)$ as x varies throughout D is called the *range* of the function.

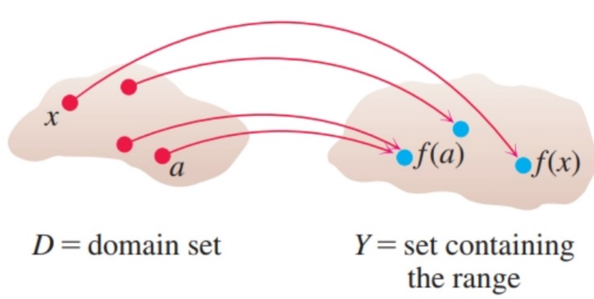

Figure 1.2

Note. The fact that each number in the domain of f is assigned a *unique* number in the range of f, implies that the graph of f will satisfy the *vertical line test*. That is, a vertical line will intersect the graph of a function in at most one point.

Note. When finding domains, we usually look for *bad* values of x which must be excluded from the domain. Primary among the restrictions of the mathematical world is the fact that **you cannot now, nor in the future, divide by 0**!!! I will not divide by 0, my colleagues will not divide by 0, and you will not divide by 0. In addition, in this class, we restrict our attention to real numbers. Therefore **we will not take square roots of negative numbers**! This is a restriction which is rather different from the restriction against dividing by 0. There are areas of math "out there" in which people take square roots of negatives (in the world of complex numbers—in fact, in this world you can also take a logarithm of negatives and an inverse sine of numbers greater than 1). The ETSU class Complex Variables (MATH 4337/5337) deals with this topic and it could be described as "the calculus of complex numbers." I have notes online for the Complex Variable class, as well as some Complex Variables videos.

Example. Exercise 1.1.4.

Definition. If f is a function with domain D, its *graph* consists of the points in the Cartesian plane whose coordinates are the input-output pairs of f. In set notation, the graph is $\{(x, f(x)) \mid x \in D\}$.

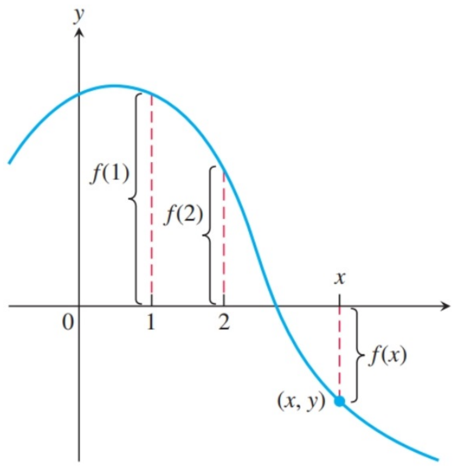

Figure 1.4

Note. In this class, we will often define a function *piecewise*. That is, instead of giving a single formula for a function, we will give several formula which define the function piecewise over certain points or intervals. Though there may be several pieces, we will have only one function.

Example. Example 1.1.4. Consider

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1. \end{cases}$$

This function is *piecewise defined*. The function comes in three pieces (but there is only **one** function; its name is "f"). You know how to graph lines and parabolas (see my online Precalculus 1 (Algebra) [MATH 1710] on 1.3. Lines for a review of graphing lines, and on 3.3. Quadratic Functions and Their Properties for a review of graphing parabolas). So we simply graph the line $y = -x$ for $x < 0$, graph the parabola $y = x^2$ for $0 \leq x \leq 1$, and graph the line $x = 1$ for $x > 1$. These pieces give the graph of $y = f(x)$. The graph of this function is:

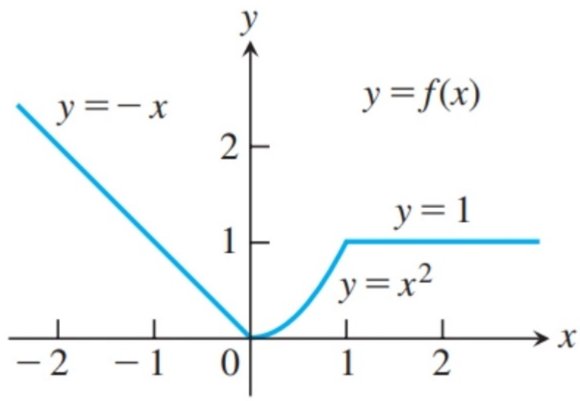

Figure 1.9

Example. Exercise 1.1.26.

Example. Two functions which will make useful examples in our study of one-sided limits (Section 2.4) are the *greatest integer function* $f(x) = \lfloor x \rfloor$ and the *least integer function* $g(x) = \lceil x \rceil$:

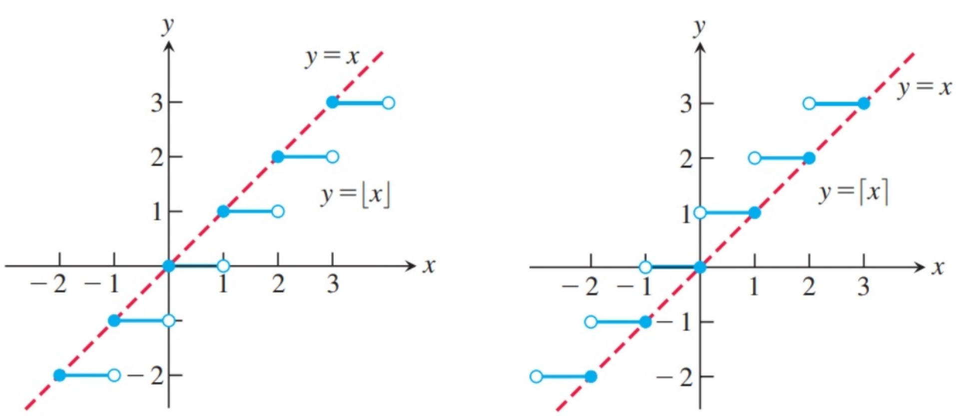

Figure 1.10 and Figure 1.11

Definition. Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$, then f is said to be *increasing* on I.

2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be *decreasing* on I.

Note. It is difficult to tell whether a function is increasing or decreasing unless you have the graph of the function. In Section 4.3 we will have a method to determine the increasing/decreasing properties of a function and then *use* these properties to create a graph.

Definition. A function $y = f(x)$ is an

- *even function of* x if $f(-x) = f(x)$,
- *odd function of* x if $f(-x) = -f(x)$,

for every x in the function's domain.

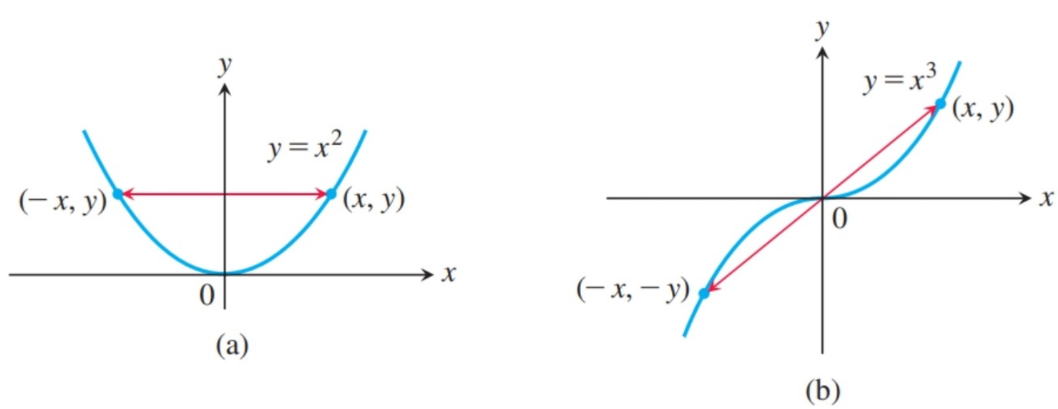

Figure 1.12

Definition. The graph of an even function is said to be *symmetric about the y-axis*. The graph of an odd function is said to be *symmetric about the origin*.

Example. Exercise 1.1.58.

Definition. A *linear function* is a function of the form $f(x) = mx + b$, where m and b are constants. The constant m is the *slope* of the linear function and the larger m is in magnitude, the steeper is the graph of f.

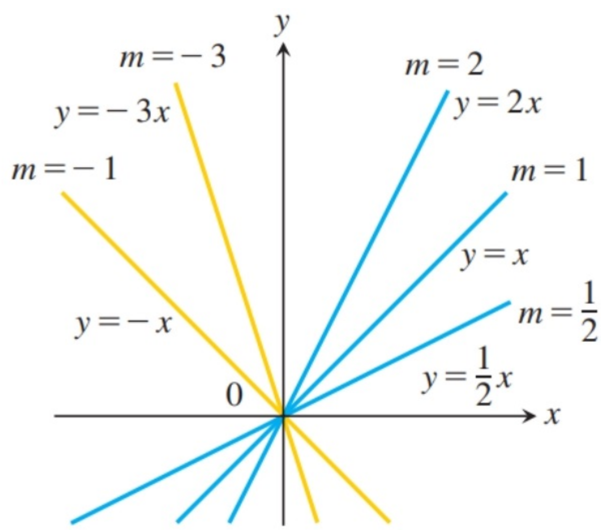

Figure 1.14(a)

Definition. A *power function* is a function of the form $f(x) = x^a$, where a is a constant.

Note. Figures 1.15, 1.16, and 1.17 give the graphs of power functions for several different values of a.

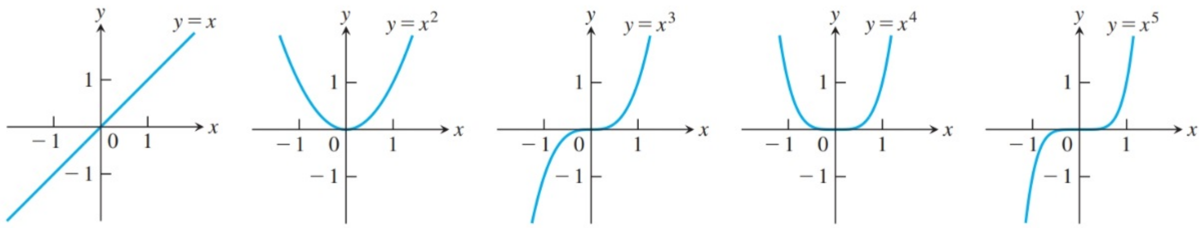

FIGURE 1.15 Graphs of $f(x) = x^n$, $n = 1, 2, 3, 4, 5$, defined for $-\infty < x < \infty$.

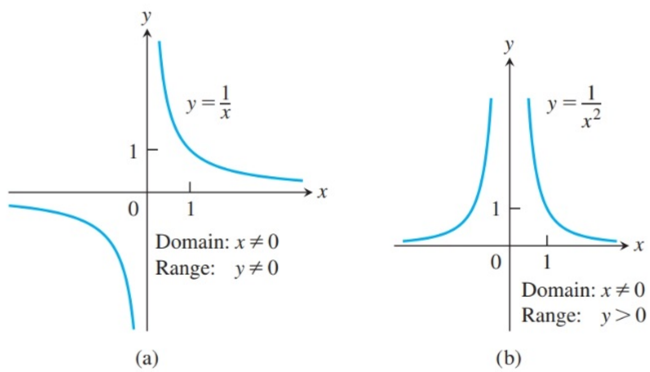

FIGURE 1.16 Graphs of the power functions $f(x) = x^a$. (a) $a = -1$, (b) $a = -2$.

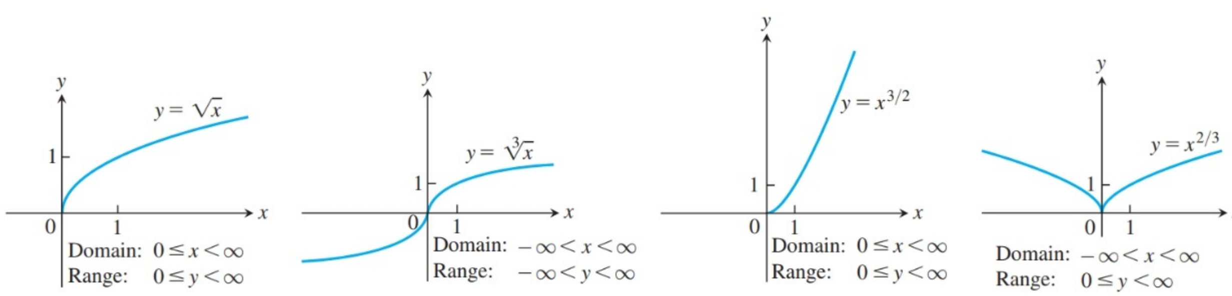

FIGURE 1.17 Graphs of the power functions $f(x) = x^a$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

Definition. A *polynomial function* is a function of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer called the *degree* of the polynomial. The constants $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are the *coefficients* of the polynomial.

Note. Figure 1.18 shows the graphs of several polynomials. Notice that the graphs of the polynomials sometime increase, sometimes decrease, and are "smooth." The domain is all real numbers.

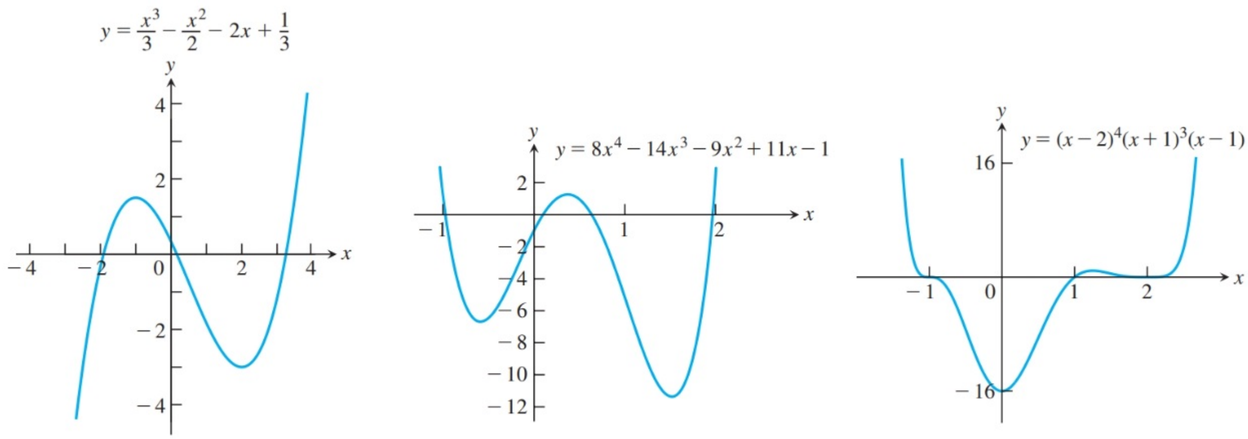

Figure 1.18

Definition. A *rational function* is a quotient (or ratio) of polynomials $f(x) = p(x)/q(x)$ where p and q are polynomials.

Note. Figure 1.19 shows the graphs of several rational functions. Notice that the graphs of the rational functions sometime increase, sometimes decrease, and are "smooth." They can have horizontal asymptotes, vertical asymptotes, and "oblique" asymptotes. The domain is all real numbers where the denominator is nonzero.

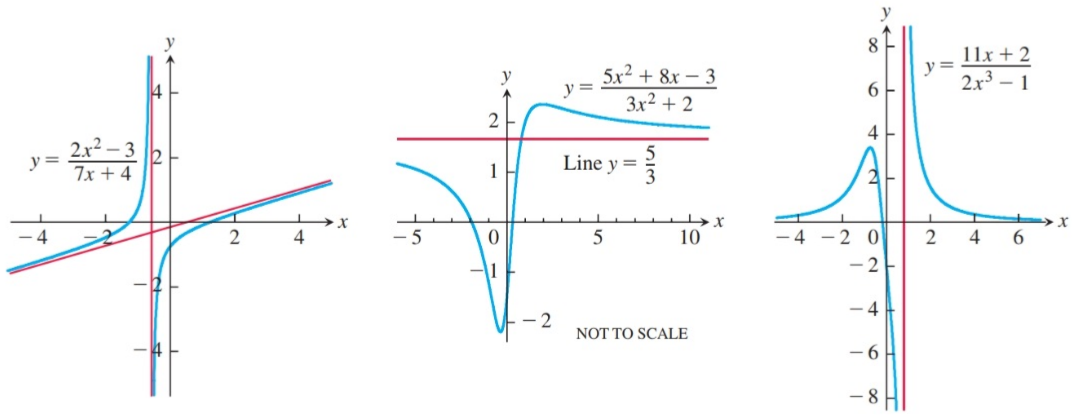

Figure 1.19

Note. We will learn how to graph polynomials and rational functions in Chapter 4. These functions lie in the broader class of functions called algebraic functions. Any function constructed from polynomials using the algebraic operations of addition, subtraction, multiplication, division, and taking roots is an *algebraic function*. This class is in contrast to nonalgebraic functions, or *transcendental functions*, such as logarithms, exponentials, and trigonometric functions.

Definition. A function of the form $f(x) = a^x$, where $a > 0$ and $a \neq 1$, is an *exponential function*. The inverse function of an exponential function is a *logarithmic function*, so $y = \log_a x$ if and only if $a^y = x$.

Note. Figures 1.22 and 1.23 show the graphs of several exponential functions and logarithmic functions for different values of a. Exponential functions have $y = 0$ as a horizontal asymptote and their domain is all real numbers. Logarithmic functions have $x = 0$ as a vertical asymptote and their domain is all positive real numbers.

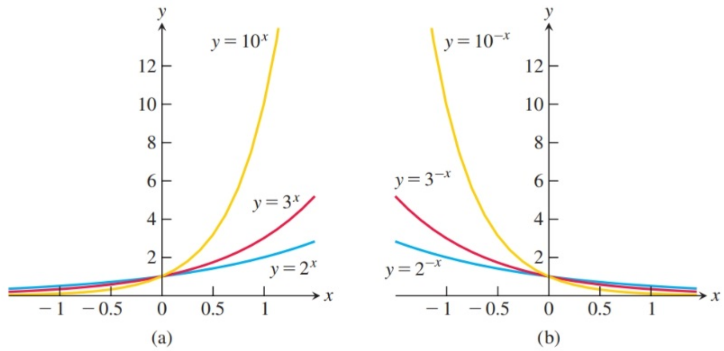

FIGURE 1.22 Graphs of exponential functions.

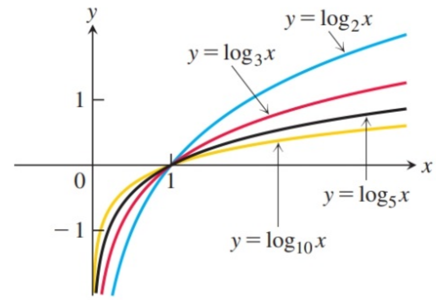

FIGURE 1.23 Graphs of four logarithmic functions.

Examples. Exercise 1.1.68 and Exercise 1.1.76.

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