Chapter 1. Functions

1.3. Trigonometric Functions

Note. In this section we give a quick review of the material of Precalculus 2 (Trigonometry) [MATH 1720]. For more details, see my online notes for Precalculus 2.

Definition. The number of *radians* in the central angle $A'CB'$ within a circle of radius r is defined as the number of "radius units" contained in the arc s subtended by the central angle. With the central angle measuring θ radians, this means $\theta = s/r$ or $s = r\theta$.

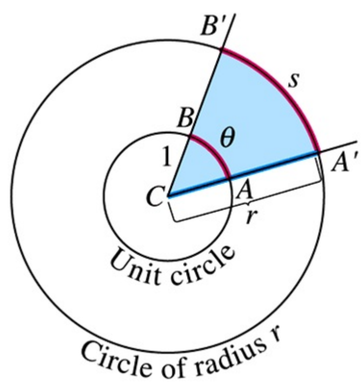


Figure 1.36

Note. One complete revolution of the unit circle is 360° or 2π radians. Therefore π radians $= 180°$ and

$$1 \text{ radian } = \frac{180°}{\pi} \approx 57.3° \text{ or } 1° = \frac{\pi}{180} \approx 0.017 \text{ radians.}$$

Radians are a unitless measure of angles and we need not write "radians" (though we often will). Degrees are not unitless and if we measure angles in degrees then we must include the degrees symbol ∘. In the calculus classes, we will only rarely use degrees to measure angles and will almost exclusively use radians.

Example. Exercise 1.3.2.

Definition. An angle in the xy-plane is in *standard position* if its vertex lies at the origin and its initial ray lies along the positive x-axis. Angles measured counterclockwise from the positive x-axis are assigned positive measures; angles measured clockwise are assigned negative measures.

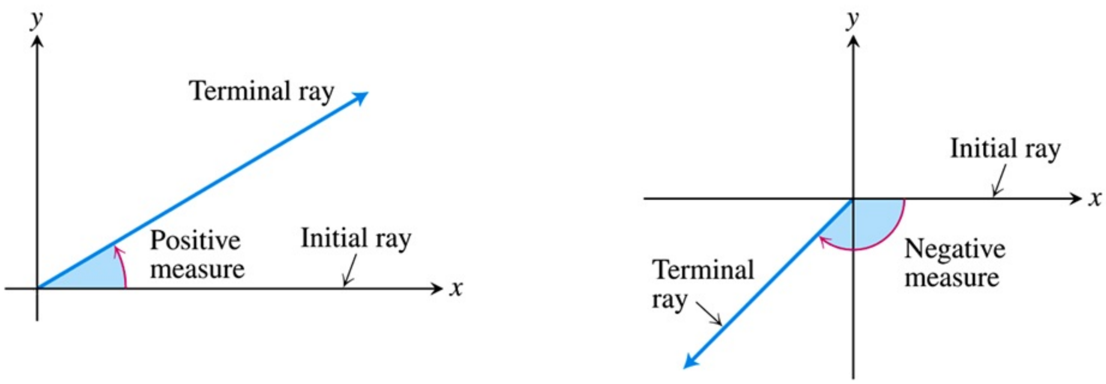


Figure 1.37

Note. We can define the six trigonometric functions for acute angles using a right triangle:

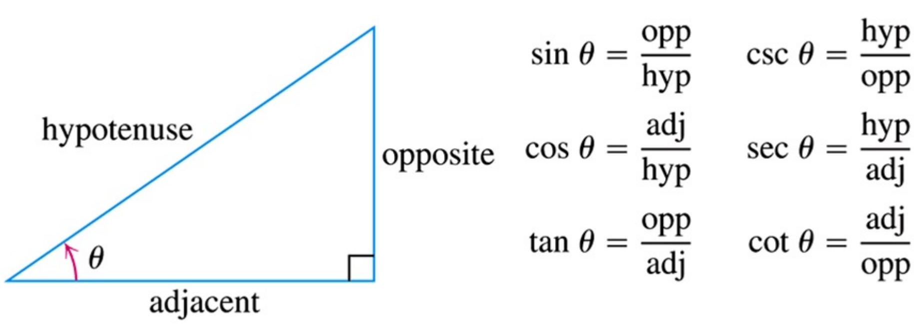


Figure 1.39

Definition. We define the six trigonometric functions for any angle θ by first placing the angle in standard position in a circle of radius r. Then define the trigonometric functions in terms of the coordinates of the point $P(x, y)$ where the angle's terminal ray intersects the circle as follows:

$$\sin\theta = \frac{y}{r} \quad \cos\theta = \frac{x}{r}$$
$$\sec\theta = \frac{r}{x} \quad \csc\theta = \frac{r}{y}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

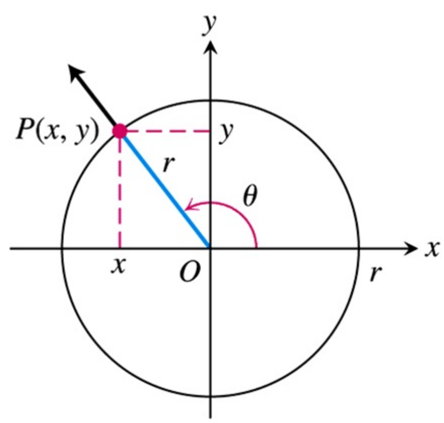


Figure 1.40

Note. By definition, we immediately have the trig identities:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \cot\theta = \frac{1}{\tan\theta}$$
$$\sec\theta = \frac{1}{\cos\theta} \quad \csc\theta = \frac{1}{\sin\theta}$$

Note. Based on the special 30-60-90 and 45-45-90 right triangles, we can deduce the following trig functions for the "special angles" $30° = \pi/6$, $45° = \pi/4$, and $60° = \pi/3$:

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$
$$\sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$
$$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

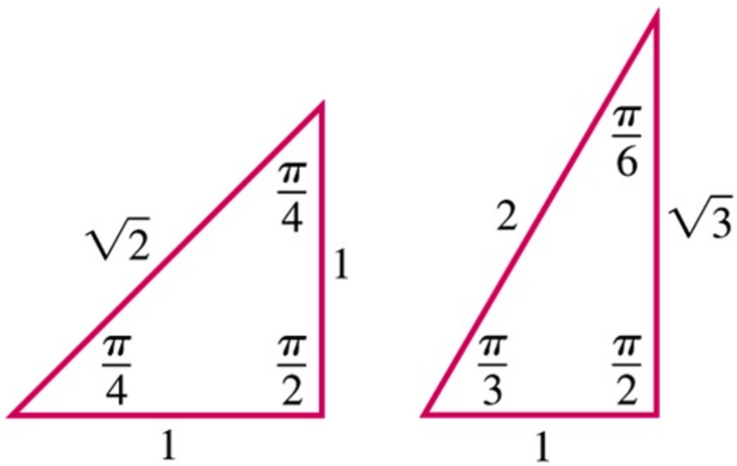


Figure 1.41

Example. Exercise 1.3.6.

Definition. A function $f(x)$ is *periodic* if there is a positive number p such that $f(x+p) = f(x)$ for every value of x. The smallest value of p is the *period* of f.

Note. The periods of sine, cosine, secant, and cosecant are each 2π. The periods of tangent and cotangent are both π. This leads to the trig identities:

$$\sin(x+2\pi) = \sin x \quad \cos(x+2\pi) = \cos x$$

$$\sec(x+2\pi) = \sec x \quad \csc(x+2\pi) = \csc x$$

$$\tan(x+\pi) = \tan x \quad \cot(x+\pi) = \cot x$$

Note. The graphs of the six trigonometric functions are as follows (the shading indicates a single period):

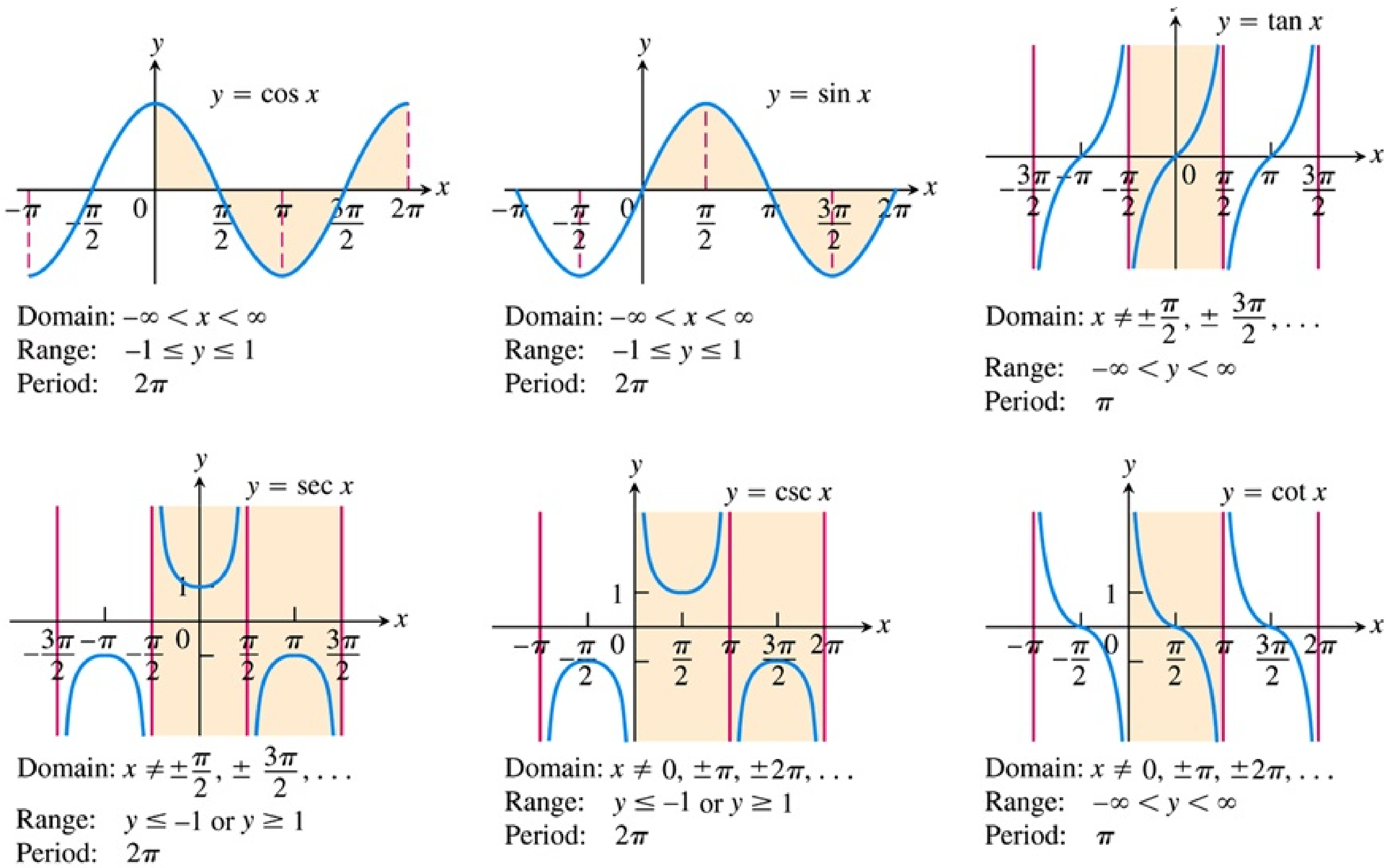


Figure 1.41

Note. From the observed symmetries of the graphs, we see that cosine and secant are even functions ($\cos(-x) = \cos x$ and $\sec(-x) = \sec x$) and that sine, tangent, cosecant, and cotangent are odd functions ($\sin(-x) = -\sin x$, $\tan(-x) = -\tan x$, $\csc(-x) = -\csc x$, $\cot(-x) = -\cot x$).

Theorem 1.3.A. $\cos^2\theta + \sin^2\theta = 1$.

Note. The proof is based on the Pythagorean Theorem (and actually *is* the Pythagorean Theorem stated in terms of trig functions. See Figure 1.45.

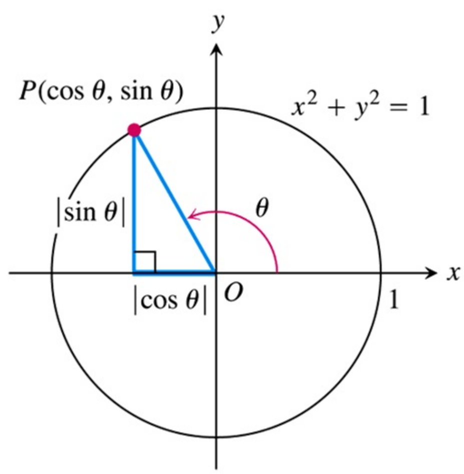


Figure 1.45

Note. Based on the above identity, we immediately have:

$$1 + \tan^2\theta = \sec^2\theta \text{ and } 1 + \cot^2\theta = \csc^2\theta.$$

Note. Other common trig identities include:

- Sum and Difference Formulas: $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$, $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$, $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$.
- Double Angle Formulas: $\cos 2\theta = \cos^2\theta - \sin^2\theta$, $\sin 2\theta = 2\sin\theta\cos\theta$, $\tan 2\theta = \dfrac{2\tan^2\theta}{1-\tan^2\theta}$.
- Half-Angle Formulas: $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$, $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$, $\tan^2\theta = \dfrac{1-\cos 2\theta}{1+\cos 2\theta}$.

Note. For proofs of the sum and difference see my online notes for Precalculus 2 (Trigonometry) [MATH 1720] on 6.4. Sum and Difference Formulas. For proofs of the double and half angle formulas, see 6.5. Double-Angle and Half-Angle Formulas.

Example. Exercise 1.3.31. This example illustrates why cosine is called "**co**sine."

Theorem 1.3.B. The Law of Cosines. If a, b, and c are sides of a triangle ABC and if C is the angle opposite c, then

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Note. For a proof of the Law of Cosines see my online notes for Precalculus 2 (Trigonometry) [MATH 1720] on 7.3. The Law of Cosines.

Theorem 1.3.C. The Law of Sines. If a, b, and c are sides of a triangle ABC with angle A opposite a, angle B opposite b, and angle C opposite c, then

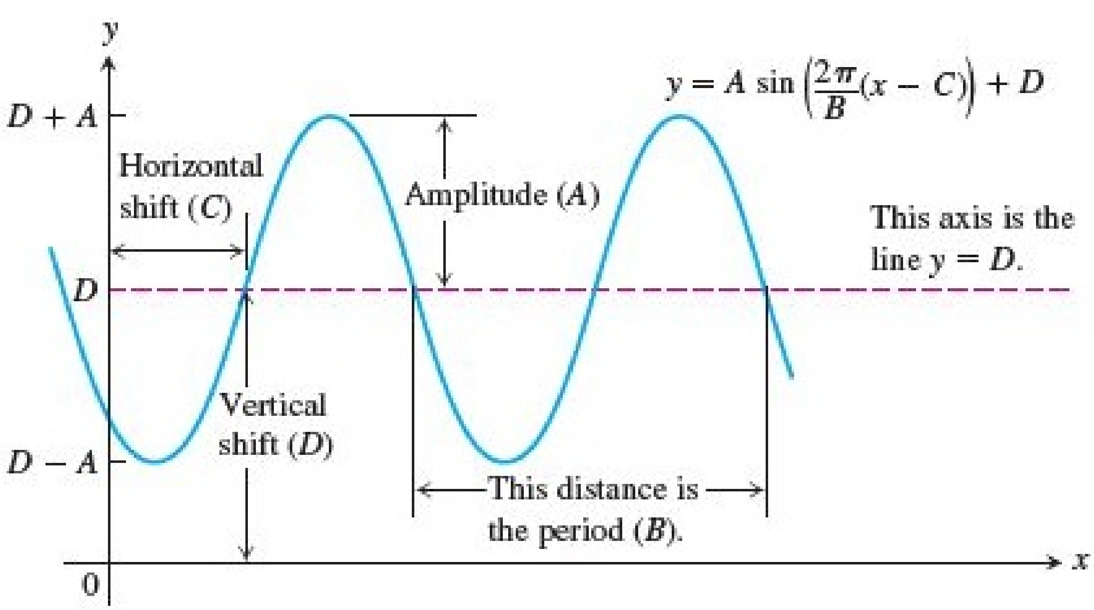
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Note. For a proof of the Law of Sines see my online notes for Precalculus 2 (Trigonometry) [MATH 1720] on 7.2. The Law of Sines.

Note. The general sine function is of the form

$$f(x) = A\sin\left(\frac{2\pi}{B}(x - C)\right) + D.$$

The *amplitude* is $|A|$, the *period* is $|B|$, the *horizontal shift* is C, and the *vertical shift* is D.



Example. Example 1.3.A: For any angle θ measured in radians, we have

$$-|\theta| \leq \sin\theta \leq |\theta| \text{ and } -|\theta| \leq 1 - \cos\theta \leq |\theta|.$$

Example. Exercise 1.3.68.

Revised: 8/29/2020