Chapter 1. Functions

1.5. Exponential Functions

Note. In this section we give a quick review of some material from Precalculus 1 (Algebra) [MATH 1710]. For more details, see my online Precalculus 1 notes on 5.3. Exponential Functions. Exponential functions are involved in several applications, including population growth, the spread of disease, continuously compounded interest, and radioactive decay. In Chapter 7 we will use calculus to define logarithmic and exponential functions. In this section and the next, we informally deal with these functions, as you did in high school or Precalculus.

Definition. If $a \neq 1$ is a positive constant, the function $f(x) = a^x$ is the *exponential function with base a.*

Examples. Exercise 1.5.2(a) and Exercise 1.5.8(a).

Note. If $x = n \neq 0$ is an integer, then

$$a^x = a^n = \underbrace{(a)(a)(a)\cdots(a)}_{n \text{ factors}} \text{ for } n > 0, \text{ and}$$
$$a^x = \underbrace{(1/a)(1/a)(1/a)\cdots(1/a)}_{|n| \text{ factors}} \text{ for } n < 0 \text{ and } a \neq 0.$$

If $x = p/q$ is a rational number (a ratio of integers), then $a^x = a^{p/q} = \sqrt[q]{a^p}$. If x is irrational, then the decimal representation of x has no terminating or repeating pattern. We could use rational approximations of x to approximate a^x. For example, $\sqrt{3} \approx 1.732050808$, and we could approximate $2^{\sqrt{3}}$ by using more and more decimals of $\sqrt{3}$:

$$2^1,\ 2^{1.7},\ 2^{1.73},\ 2^{1.732},\ 2^{1.7320},\ 2^{1.73205},\ldots$$

We would then look for the *limit* of the sequence. It is the Axiom of Completeness in the definition of the real numbers that guarantees that this sequence actually has a limit (see Appendix A.6. Theory of the Real Numbers where the Axiom of Completeness is described in terms of an airplane taking off).

Note. We have already tried to graph $y = 3^x$ in Exercise 1.5.2(a). Figure 1.53 gives the graphs of this function, along with several others. On the right, the functions $y = 10^{-x}$, $y = 3^{-x}$, and $y = 2^{-x}$ are graphed. These are better thought of as $y = (1/10)^x$, $y = (1/3)^x$, and $y = (1/2)^x$, respectively, so that these are exponential functions of the form $f(x) = a^x$ where $a = 1/10$, $a = 1/3$, and $a = 1/2$, respectively. The pattern is that for $a > 1$ an exponential function is increasing (and it increases faster when a is large), and for $0 < a < 1$ and exponential function is decreasing (and it decreases faster when a is small).

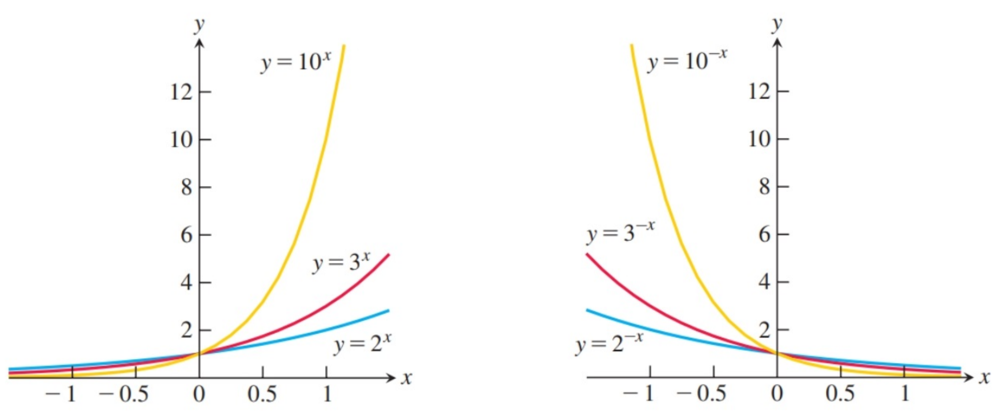

Figure 1.53

Theorem 1.5.A. Rules for Exponents.

If $a > 0$ and $b > 0$, the following rules old true for all real numbers x and y.

1. $a^x a^y = a^{x+y}$

2. $\dfrac{a^x}{a^y} = a^{x-y}$

3. $(a^x)^y = (a^y)^x = a^{xy}$

4. $a^x b^x = (ab)^x$

5. $\dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x$

Note. A proof of Theorem 1.5.A will be given in Section 4.2. The Mean Value Theorem, where is is proved for $a = b = e$ (from which the more general result of Theorem 1.5.A follows).

Example. Exercise 1.5.12.

Definition. The number e (to be formally defined in Chapter 7) is an irrational number which is approximately

$$e \approx 2.7182818284590459.$$

An exponential function with base e is called the *natural exponential function.*

Note. Since $2 < e < 3$, we expect the graph of the natural exponential function to lie between the exponential functions 2^x and 3^x. This is illustrated in Figure 1.54, where a line tangent to the graph of the exponential function at $x = 0$ is given (notice that the slope of such a line is $m = 1$ when we consider $y = e^x$; this idea will arise again in Section 3.3. Differentiation Rules, see Figure 3.13).

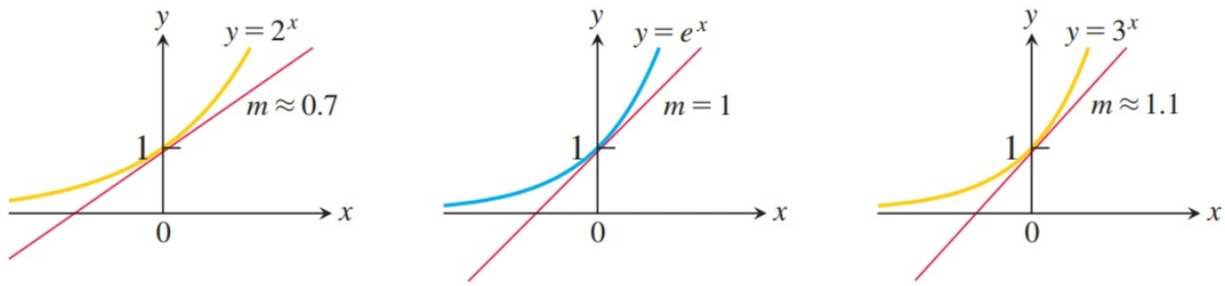

Figure 1.54

Note. The exponential functions $y = y_0 e^{kx}$, where k is a nonzero constant, are frequently used for modeling exponential growth or decay. The function $y = y_0 e^{kt}$ is a model for exponential growth if $k > 0$ and a model for exponential decay if $k < 0$.

Examples. Exercise 1.5.30(a) and Example 1.5.4.

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