Chapter 1. Functions

1.6. Inverse Functions and Logarithms

Note. In this section we give a review of some more material from Precalculus 1 (Algebra) [MATH 1710]. For more details, see my online Precalculus 1 notes on 5.2. One-to-One Functions; Inverse Functions, 5.4. Logarithmic Functions, and 5.5. Properties of Logarithms.

Definition. A function $f(x)$ is *one-to-one* on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

Note. A function $= f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once. This is called the *Horizontal Line Test*. See Figure 1.56.

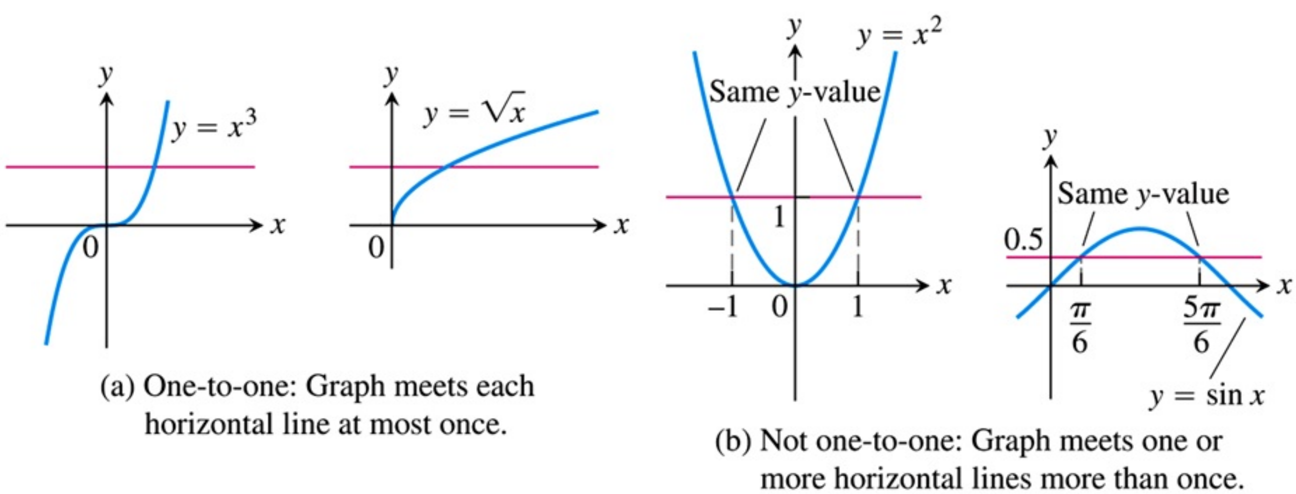


Figure 1.56

Example. Exercise 1.6.10.

Definition. Suppose that f is a one-to-one function on a domain D with range R. The *inverse function* f^{-1} is defined by

$$f^{-1}(b) = a \text{ if } f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

Note. In terms of graphs, the graph of an inverse function can be produced from the graph of the function itself by interchanging x and y values. This means that the graphs of f and f^{-1} will be mirror images of each other with respect to the line $y = x$. See Figure 1.57(c).

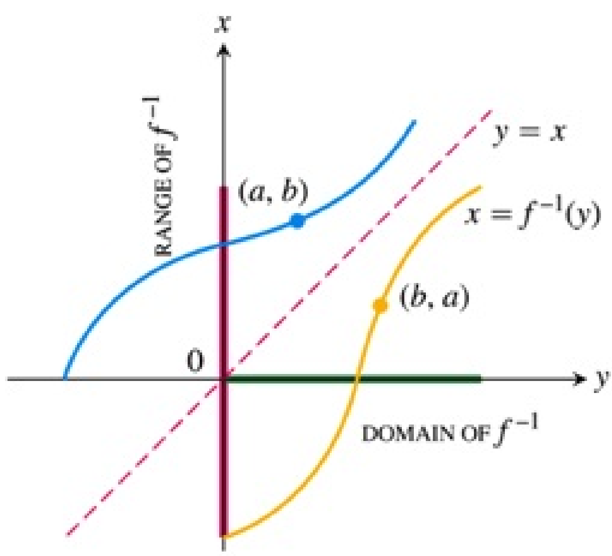


Figure 1.57(c)

Note. The process of passing from f to f^{-1} can be summarized as a two-step procedure.

1. Solve the equation $y = f(x)$ for x. This gives a formula $x = f^{-1}(y)$ where x is expressed as a function of y.

2. Interchange x and y, obtaining a formula $y = f^{-1}(x)$ where f^{-1} is expressed in the conventional format with x as the independent variable and y as the dependent variable.

Example. Example 1.6.4. Find the inverse of the function $y = x^2$, $x \geq 0$. See Figure 1.59.

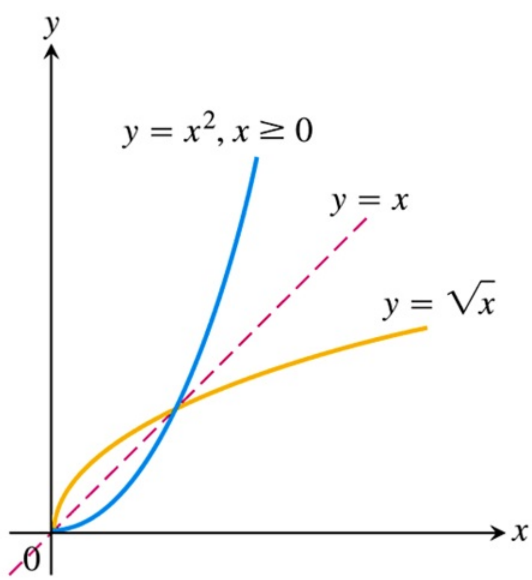


Figure 1.59

Example. Exercise 1.6.22.

Definition. The *logarithm function with base* a, $y = \log_a x$, is the inverse of the base a exponential function $y = a^x$ ($a > 0$, $a \neq 1$).

Note. The domain of $\log_a x$ is $(0, \infty)$ (the range of a^x) and the range of $\log_a x$ is $(-\infty, \infty)$ (the domain of a^x). When $a = 10$, $\log_a x = \log_{10} x$ is called the common logarithm function, sometimes denoted $\log x$. When $a = e$, $\log_a x = \log_e x$ is called the natural logarithm function, usually denoted $\ln x$ (sometimes "$\log x$" denotes the natural logarithm, but not in our text). See Figure 1.60.

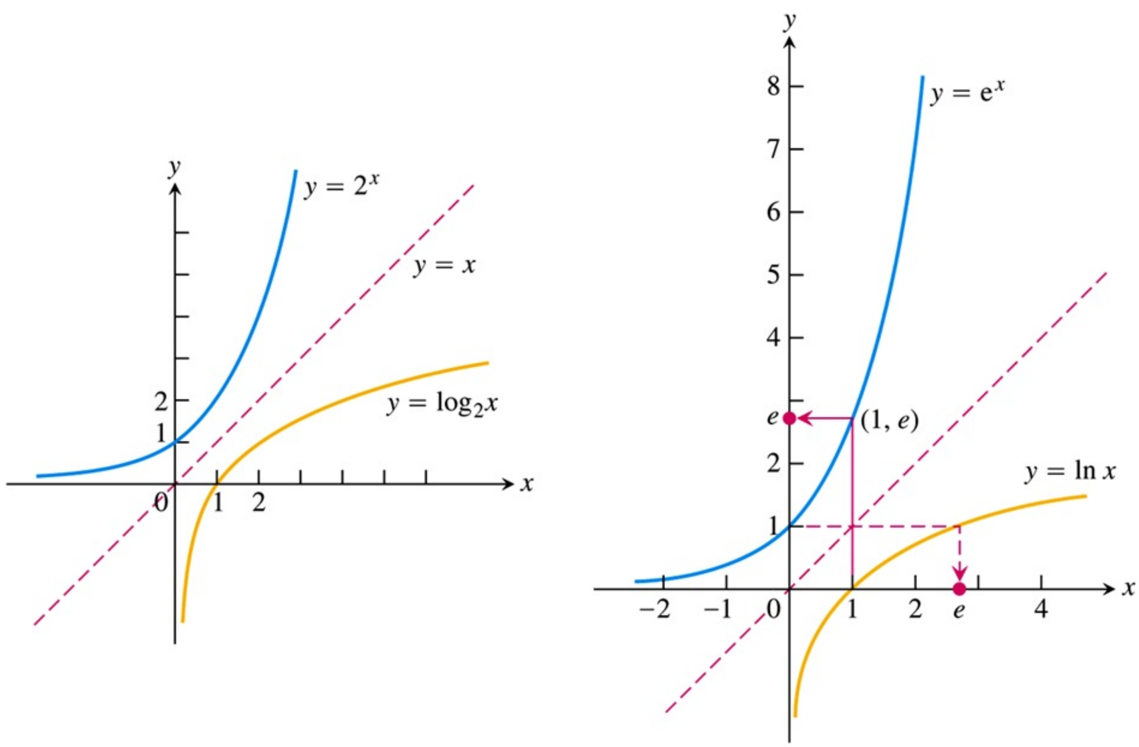


Figure 1.60

Theorem 1.6.1. Algebraic Properties of the Natural Logarithm.

For any numbers $b > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule*: $\ln bx = \ln b + \ln x$
2. *Quotient Rule*: $\ln \dfrac{b}{x} = \ln b - \ln x$
3. *Reciprocal Rule*: $\ln \dfrac{1}{x} = -\ln x$
4. *Power Rule*: $\ln x^r = r \ln x$

Example. Exercise 1.6.44.

Note. The inverse properties of a^x and $\log_a x$ are:

1. Base a: $a^{\log_a x} = x$, $\log_a a^x = x$
2. Base e: $e^{\ln x} = x$, $\ln e^x = x$

Every exponential function is a power of the natural exponential function: $a^x = e^{x \ln a}$. Every logarithm function is a constant multiple of the natural logarithm function (this is the "Change of Base Formula"): $\log_a x = \dfrac{\ln x}{\ln a}$. These last two results imply that every logarithmic and exponential function can be based on the natural log and exponential. In fact, your calculator performs all such computations using the natural functions and then converts the answer into the appropriate base.

Example. Exercise 1.6.54.

Example. Example 1.6.7.

Note. None of the six trigonometric functions is one-to-one. Therefore (as with the function $f(x) = x^2$), we *restrict the domain* of the function to create a new function which *is* one-to-one and then find the inverse of that modified function. In each of the six cases, we will keep the angles between 0 and $\pi/2$ (the acute angles) and then include more angles as given below. For example, with the sine function, we restrict the domain to $[-\pi/2, \pi/2]$ (producing a one-to-one function that takes on all values in the range of the sine function) and then find the inverse of this revised function and define the inverse as the *inverse sine function*, $\arcsin x = \sin^{-1} x$ (this is definitely **not to be confused with** the *reciprocal* of the sine function... which is the cosecant function):

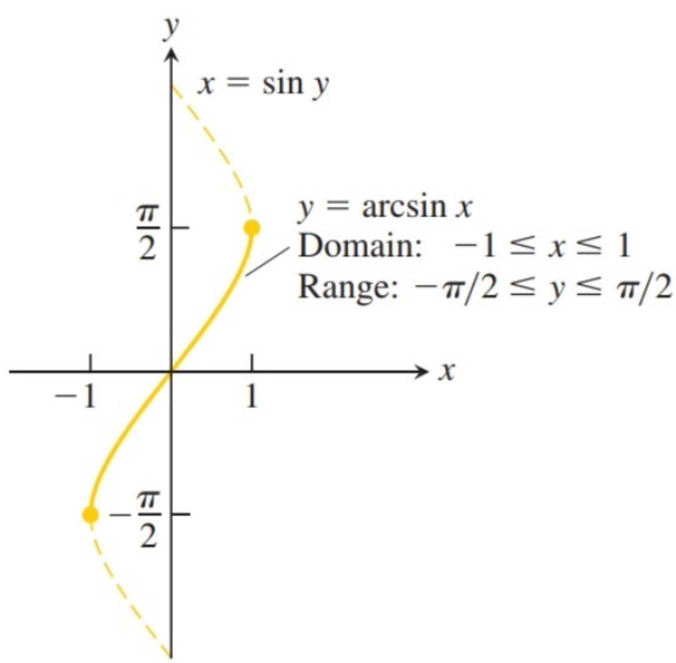
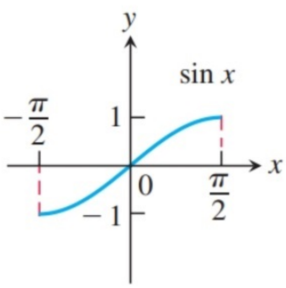
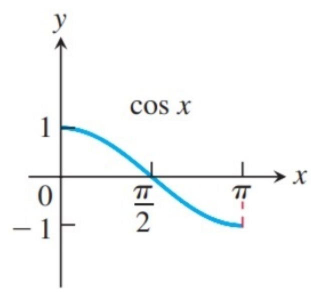


Figure 1.62

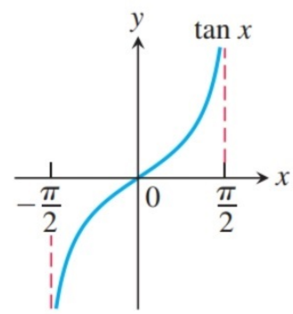
Definition. We restrict the domains of the six trig functions in order to make a new function which is one-to-one as follows:



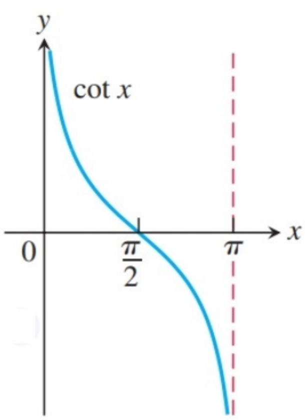
$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$



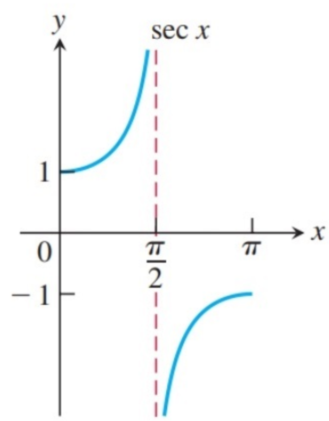
$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$



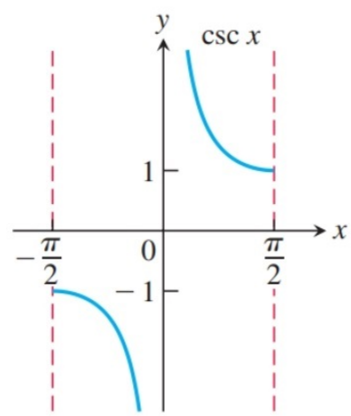
$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$

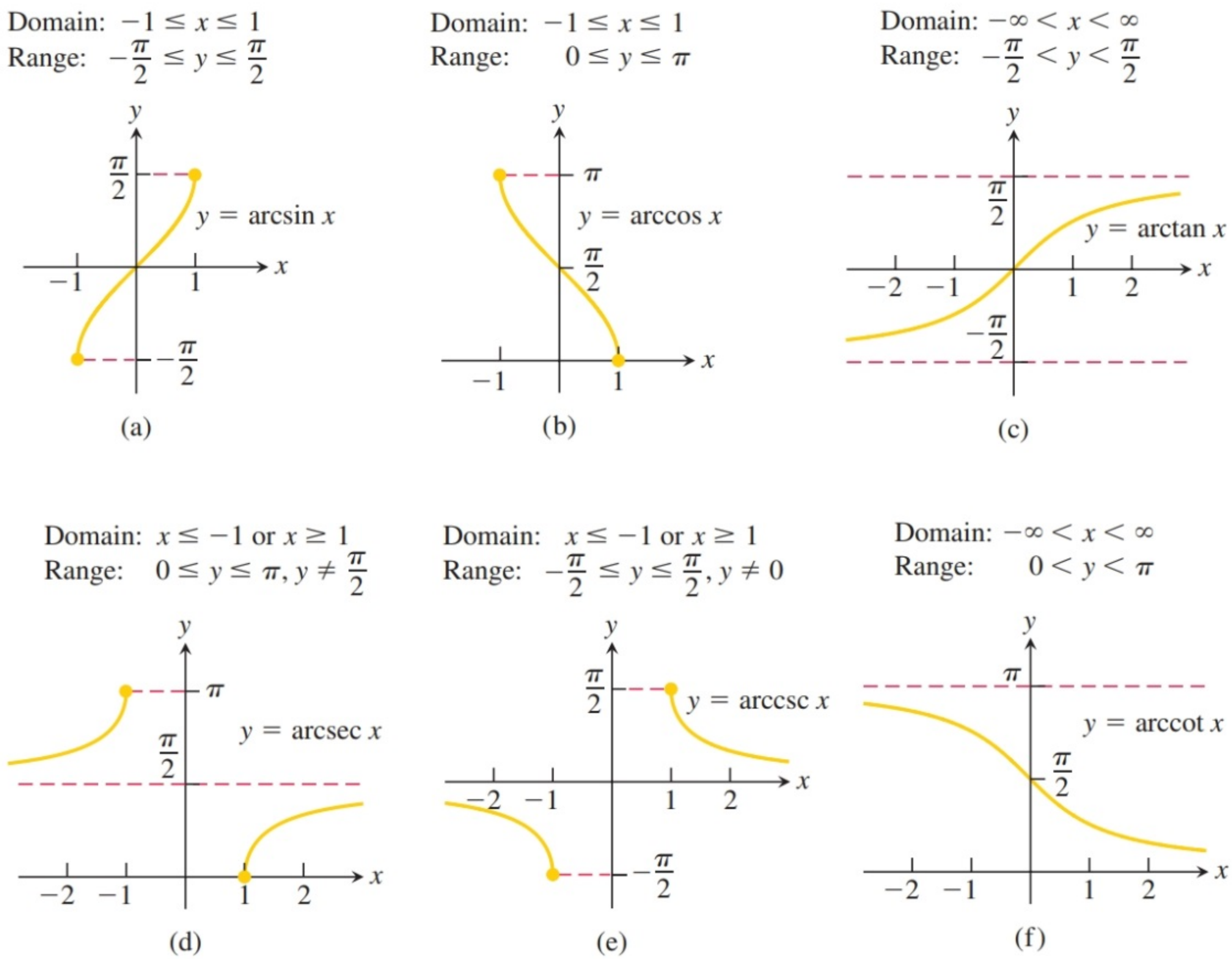


$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$



$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$

We then have the inverse trig functions (which are, in fact, inverses of not the trigonometric functions themselves, but instead the restricted functions given above):



Definition. Specifically, we have $y = \arcsin x = \sin^{-1} x$ is the number in $[-\pi/2/\pi/2]$ for which $\sin y = x$, and $y = \arccos x = \cos^{-1} x$ is the number in $[0, \pi]$ for which $\cos y = x$.

Example. Exercise 1.6.72.

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