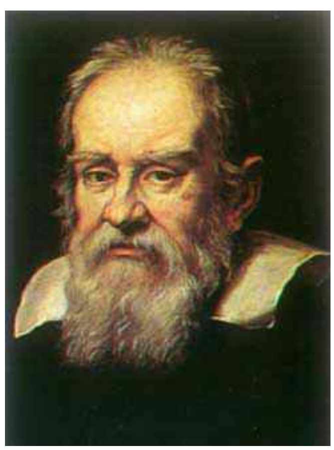
Chapter 2. Limits and Continuity

Note. You already probably have an intuitive idea of what it means for a function to be continuous. In this chapter, we develop the most fundamental idea behind calculus, that of a limit. Limits are used to define all of the topics covered in Calculus 1, 2, and 3 (. . . including continuity).

2.1. Rates of Change and Tangents to Curves

Note. Our text book motivates the study of limits in this section by considering average rates of change and discussing instantaneous rates of change. They start by discussing Galileo Galilei's (1564–1642) description of an object in free fall.



Galileo Galilei (from MacTutor History of Mathematics Archive)

Galileo stated that the distance traveled by an object in free fall is $y = 16t^2$ where y is measured in feet and t is measured in seconds after the object is released. In the first second, the object falls $16(1)^2 = 16$ feet and so its average speed over the first second is 16 ft/1 sec = 16 ft/sec. In the first two seconds, the object falls $16(2)^2 = 64$ feet and so its average speed over the first 2 seconds is 64 ft/2 sec = 32 ft/sec. Between second 1 and second 2, the object moves $64 - 16 = 48$ feet and so its average speed over this period of time is 48 ft/1 sec = 48 ft/sec; so the object is accelerating. More generally, if the distance an object has traveled is $f(t)$ at time t, then the objects *average speed* during the time interval $[t_1, t_2]$ is found by dividing the distance traveled (which is $f(t_2) - f(t_1)$) by the elapsed time (which is $t_2 - t_1$):

$$\text{Average speed over } [t_1, t_2] = \frac{\text{distance traveled}}{\text{elapsed time}} = \frac{\Delta y}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}.$$

Here we have used the Greek letter Δ (delta) to indicate a difference in values.

Example. Example 2.1.1. (We refer to examples and exercises from the book by numbering them as chapter.section.example-number or chapter.section.exercise-number.)

Note. If we calculate the average speed of a free falling object over the time interval $[t_0, t_0 + h]$ (a time interval of length $h = \Delta t$) where $y = 16t^2$, then we have

$$\text{Average speed over } [t_0, t_0 + h] = \frac{16(t_0 + h)^2 - 16t_0^2}{(t_0 + h) - t_0}$$

$$= \frac{16(t_0 + h)^2 - 16t_0^2}{h} = \frac{16(t_0 + h)^2 - 16t_0^2}{\Delta t}.$$

We are now interested in the instantaneous speed of the object. If we take the length of the time interval to be very small, then this should give a good *approximation* of the instantaneous speed at time t_0. Of course we cannot divide by $h = \Delta t = 0$;

you cannot now, nor at any point in your future mathematical career, divide by zero! So we now make a table based on $t_0 = 1$ sec and $t_0 = 2$ sec, where we let $h = \Delta t$ take on several "small" values. See Table 2.1.

TABLE 2.1 **Average speeds over short time intervals** $[t_0, t_0 + h]$

$$\text{Average speed: } \frac{\Delta y}{\Delta t} = \frac{16(t_0 + h)^2 - 16t_0{}^2}{h}$$

Length of time interval h	**Average speed over interval of length** h **starting at** $t_0 = 1$	**Average speed over interval of length** h **starting at** $t_0 = 2$
1	48	80
0.1	33.6	65.6
0.01	32.16	64.16
0.001	32.016	64.016
0.0001	32.0016	64.0016

It seems that the average speed on intervals starting at $t_0 = 1$ are approaching the value 32 ft/sec, and on intervals starting at $t_0 = 2$ are approaching the value 64 ft/sec. With $t_0 = 1$ and $h \neq 0$ we have

$$\frac{\Delta y}{\Delta t} = \frac{16(1+h)^2 - 16(1)^2}{h} = \frac{16(1 + 2h + h^2) - 16}{h}$$
$$= \frac{32h + 16h^2}{h} = \frac{h(32 + 16h)}{h} = 32 + 16h \text{ since } h \neq 0.$$

So, indeed, when h is "small" **but not 0** then the average speed is "close to" 32 ft/sec. When $t_0 = 2$, we similarly get that $\Delta y/\Delta t = 64 + 16h$ when $h \neq 0$ and for h small (but not 0) then the average speed is close to 64 ft/sec. What we mean by "small" and "close to" is the major concept of this class and the topic we address in some detail in the next few sections.

Note. We use the above observations to motivate the following definition. We need the function f (or $y = f(x)$) to be defined on the interval $[x_1, x_2]$ and we consider the line secant to the graph of $y = f(x)$ which passes through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$. We take $h = \Delta x = x_2 - x_1$ (and largely use the symbol h to represent the change in x over an interval). See Figure 2.1.

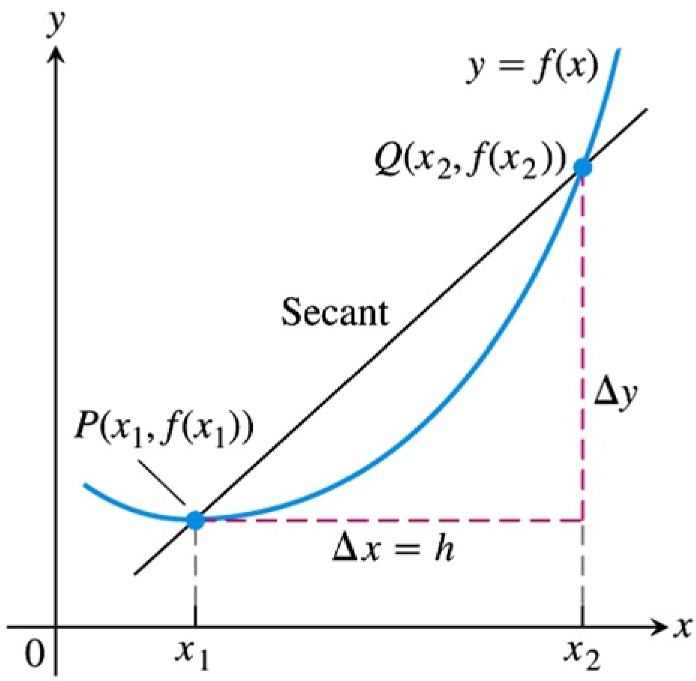


Figure 2.1

Definition. The *average rate of change* of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}$$

where $h = x_2 - x_1 \neq 0$.

Example. Exercise 2.1.4.

Note. We now informally define the *slope of a curve* at a point P on the curve. At this stage, the slope of a *line* is defined, so we use this as a starting point. We define the slope of a curve at a point P as the slope of the line tangent to the curve at point P. To find this tangent line, we approximate it by lines secant to the curve which pass through point P and another point on the curve, say point Q. Since we know two points on the secant line, P and Q, we can find the slope of the secant line. If we make point Q *really close to* point P, then the slope of the secant line should be close to the slope of the tangent line. To find the exact slope of the tangent line, requires that we take a *limit*—and limits are the topic of this chapter.

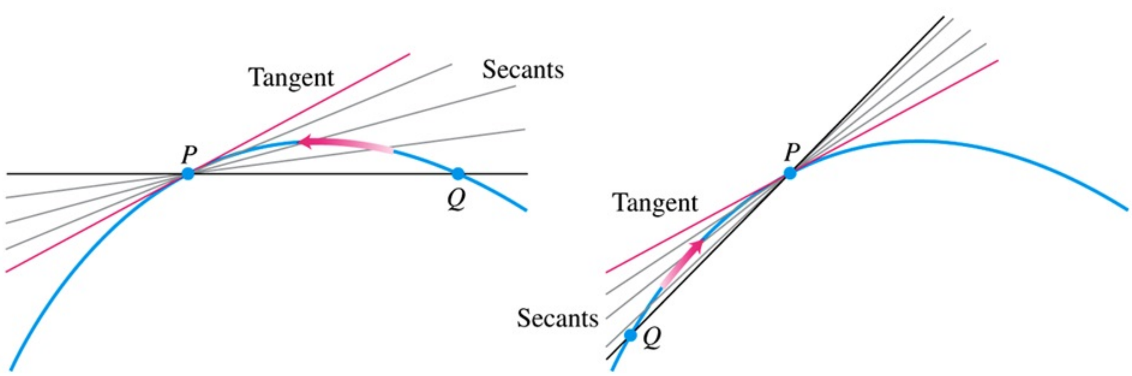


Figure 2.3

Example. Example 2.1.3.

Examples. Exercises 2.1.14, 2.1.22, 2.1.26

Revised: 6/23/2020