Chapter 2. Limits and Continuity

2.2. Limit of a Function and Limit Laws

Note. Taking our lead from the previous section and the estimation of tangent lines to a curve, we now informally discuss the idea of a limit. These notes largely follow *Thomas' Calculus*, but the approach here is very careful to respect the underlying rigorous math (which we see in the next section with the formal definition of limit).

Note. We have to be careful in our dealings with functions! Notice that $f(x) = \dfrac{(x+1)(x-1)}{x-1}$ and $h(x) = x+1$ are **NOT** the same functions! They do not even have the same domains. Therefore we *cannot* in general say $\dfrac{(x+1)(x-1)}{x-1} = x+1$. However, this equality holds if x lies in the domains of the functions. We *can* say:

$$\frac{(x+1)(x-1)}{x-1} = x+1 \textbf{ IF } x \neq 1.$$

We can also say $f(x) = h(x)$ **IF** $x \neq 1$. See Figure 2.7.

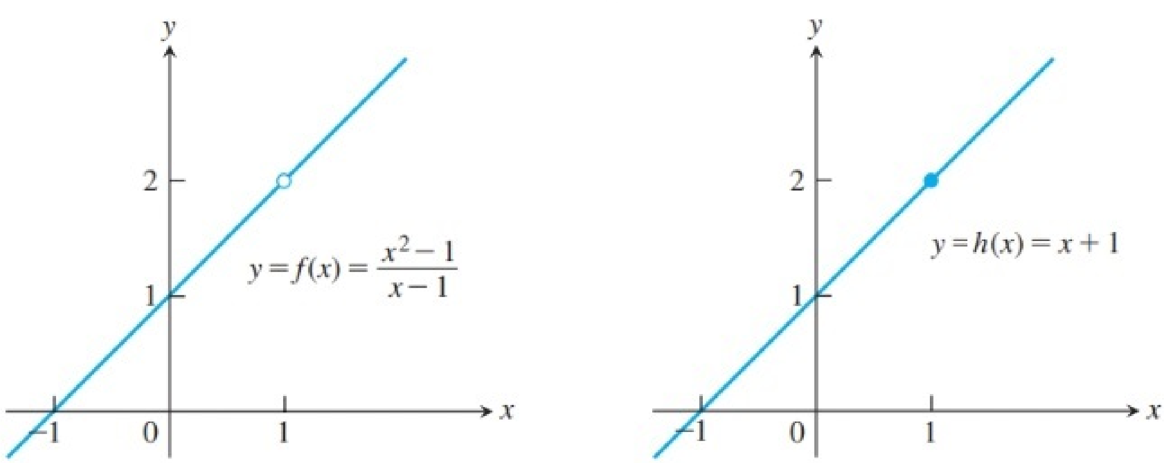

Figure 2.7. (slightly modified)

Note. We now want to talk about the behavior of $f(x) = \dfrac{(x+1)(x-1)}{x-1}$ for x "near" 1 but not equal to 1. Well, f and h are the same functions for $x \neq 1$, so the behavior of f for x near 1 but not equal to 1 is the same as the behavior of h for such values of x. In particular, when x is "really close to" 1 then $h(x)$ is "really close to" 2 (and, hence, so is $f(x)$). The only difference is what happens **at** $x = 1$: f is not defined at $x = 1$ and $h(1) = 2$. We want to say that f is *trying* to equal 2 at $x = 1$! We now give an informal definition of limit. We'll return to the attempts of function f soon...

Definition. Informal Definition of Limit.

Let $f(x)$ be defined on an open interval about c, **except possibly at c itself**. If $f(x)$ gets arbitrarily close to L for all x sufficiently close to c (but not equal to c), we say that f *approaches the limit L as x approaches c*, and we write

$$\lim_{x \to c} f(x) = L.$$

Note. The above definition is **informal** (that is, it is not mathematically rigorous) since the terms "arbitrarily close" and "sufficiently close" are not defined. Let's use this informal idea to further analyze functions f and h from above.

Example. Evaluate $\displaystyle\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{(x+1)(x-1)}{x-1}$.

Solution. From above, we see that

$$f(x) = \frac{(x+1)(x-1)}{x-1} = x + 1 \textbf{ IF } x \neq 1.$$

So $\lim_{x \to 1} f(x) = \lim_{x \to 1} \dfrac{(x+1)(x-1)}{x-1} = \lim_{x \to 1} x + 1$ if $x \neq 1$. Therefore, in words, we ask "what does $x+1$ get close to when x is close to 1?" Well, $x+1$ gets close to 2! So $\lim_{x \to 1} f(x) = \lim_{x \to 1} \dfrac{(x+1)(x-1)}{x-1} = \lim_{x \to 1} x + 1 = \boxed{2}$, even though we are not allowing x to equal 1; the important thing is that x can be made arbitrarily close to 1. Notice that this example shows that $\lim_{x \to 1} f(x) = \lim_{x \to 1} h(x)$, where $h(x) = x + 1$. That is, the limits of f and h are the same, even though the functions f and h are different (though very subtly different—they only differ at $x = 1$).

Example. Example 2.2.A. Use the above technique to evaluate $\lim_{x \to 3} \dfrac{x^2 - 4x + 3}{x - 3}$.

Note. Another very informal idea is the following:

> **Dr. Bob's Anthropomorphic Definition of Limit.**
>
> Let $f(x)$ be defined on an open interval about c, **except possibly at** c **itself**. If the graph of $y = f(x)$ *tries to pass through the point* (c, L), then we say $\lim_{x \to c} f(x) = L$. Notice that it does not matter whether the graph actually passes through the point, only that it *tries to*.

Note 2.2.A. First, we establish two fundamental limits using Dr. Bob's Anthropomorphic Definition of Limit. Consider the functions $f(x) = x$ (the identity function) and $g(x) = k$ (a constant function). Notice that function $f(x) = x$ tries to pass through the point (c, c) and function $g(x) = k$ tries to pass through the point (c, k) (in fact, both succeed in passing through these respective points, though that is irrelevant to the existence of a limit), so that both $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$

exist and are c and k, respectively. See Figure 2.9.

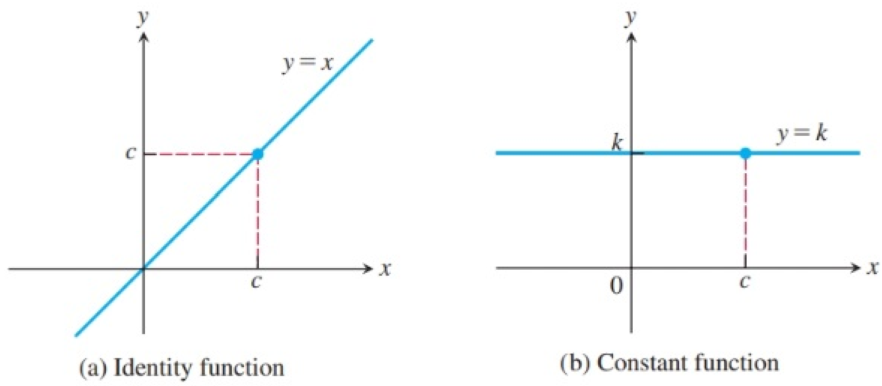

Figure 2.9.

Note 2.2.B. We have the graphs of three functions in Figure 2.8. The functions are each the same, except at $x = 1$. To illustrate what it means for a function to "try" to pass through a point, we claim that each of the three functions try to pass through the point $(1, 2)$. Function h actually succeeds in passing through the point (we will use this property to define continuity later). Both f and g fail to pass through the point $(1, 2)$, but in different ways. Function f fails to pass through the point because it is not even defined at $x = 1$. Function g fails to pass through the point $(1, 2)$ (though it tries!), and instead its graph contains the point $(1, 1)$.

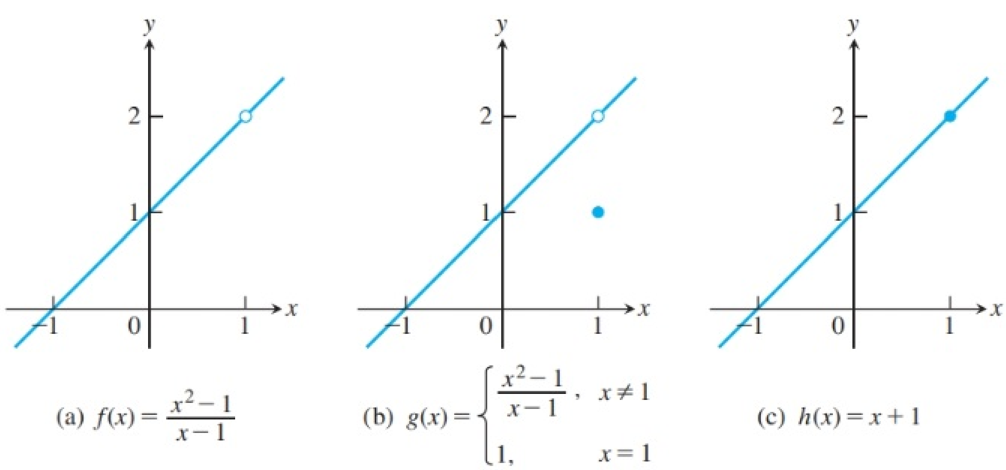

Figure 2.8.

Note. We have the graphs of three functions in Figure 2.10. We argue that the limit as x approaches 0 does not exist for any of these three functions. For the unit step function U, there is not a single point that the graph tries to pass through *from both sides*, so the limit does not exist (though we could argue that there are points that it tries to pass through from each side; we will consider one-sided limits in Section 2.4). For the function g, the graph does not get close to any particular point when x is close to 0 and so there is not a point the graph tries to pass through and the limit does not exist. For the function f, the graph oscillates wildly between -1 and 1 for x close to 0 and positive; the graph gets close to all the points on the y-axis with y coordinate between -1 and 1, but there is not single point the graph tries to pass through as x approaches 0 and the limit does not exist.

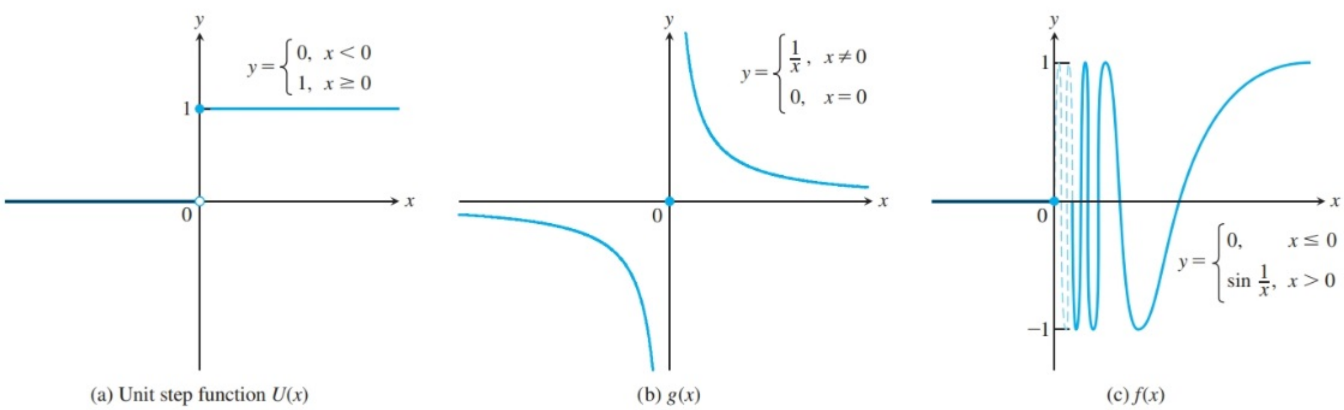

Figure 2.10.

Note. We now illustrate Dr. Bob's Anthropomorphic Definition of Limit with an example. We want to recognize when a limit exists, given the graph of a function. Remember that $\lim_{x \to c} f(x) = L$ if the graph of $y = f(x)$ tries to pass through the point (c, L); so the graph will either have a little hole in it at the point (c, L) (in the case that f tries to pass through the point but fails) or the graph will simply pass right through the point (in the case that f tries to pass through the point and succeeds).

Example. Exercise 2.2.2.

Note. The next result gives some properties of limits. We will address the proofs of some of these properties when we have a rigorous definition of limit. See also Appendix A.4.

Theorem 2.1. Limit Rules.

If L, M, c, and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule*: $\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = L + M$.

2. *Difference Rule*: $\lim_{x\to c}(f(x) - g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x) = L - M$.

3. *Constant Multiple Rule*: $\lim_{x\to c}(kf(x)) = k \lim_{x\to c} f(x) = kL$.

4. *Product Rule*: $\lim_{x\to c}(f(x)g(x)) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right) = LM$.

5. *Quotient Rule*: $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \dfrac{L}{M}$, if $\lim_{x\to c} g(x) = M \neq 0$.

6. *Power Rule*: If n is a positive integer, then $\lim_{x\to c}(f(x))^n = \left(\lim_{x\to c} f(x)\right)^n = L^n$.

7. *Root Rule*: If n is a positive integer, then $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)} = \sqrt[n]{L} = L^{1/n}$ (if n is even, we also require that $f(x) \geq 0$ on some open interval containing c, except possibly at c itself).

Example. Exercise 2.2.52.

Note. We now state two results which we can prove using Theorem 2.1. These results, along with Note 2.2.B, will give us a foundation by which we can evaluate limits using Theorem 2.1.

Theorem 2.2. Limits of Polynomials Can Be Found by Substitution.
If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ then

$$\lim_{x\to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0.$$

Theorem 2.3. Limits of Rational Functions Can Be Found by Substitution IF the Limit of the Denominator Is Not Zero.
If P and Q are polynomials and $Q(c) \neq 0$, then

$$\lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{\lim_{x\to c} P(x)}{\lim_{x\to c} Q(x)} = \frac{P(c)}{Q(c)}.$$

Examples. Exercises 2.2.14 and 2.2.18.

Note. We now state a result that allows us to evaluate most of the limits that we encounter in the first part of Calculus 1. It will allow us to apply the technique of **F**actoring, **C**anceling, and **S**ubstituting (or "FCS"). This result will follow from the definition of limit as stated in the next section.

Theorem 2.2.A. Dr. Bob's Limit Theorem.

If functions f and g satisfy $f(x) = g(x)$ for all x in an open interval containing c, except possibly c itself, then

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x),$$

provided these limits exist.

Note. Dr. Bob's Limit Theorem is a summary of what the text calls "Eliminating Zero Denominators Algebraically" and the use of "simpler fractions." The text also describes "Using Calculators and Computers to Estimate Limits" in which you plug in x-values "closer and closer" to c to estimate $\lim_{x \to c} f(x)$. However, this instructor finds the use of such estimates horribly misleading! We will skip the exercises with instructions "make a table..."

Examples. Exercises 2.2.34 and 2.2.38.

Note. The next result lets us calculate the limit of a certain function that is "sandwiched" between two functions, where we know the limit of the two functions (and they are related to the given function in a certain way). We will often use this result to establish limits involving trigonometric functions.

Theorem 2.4. Sandwich Theorem.

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

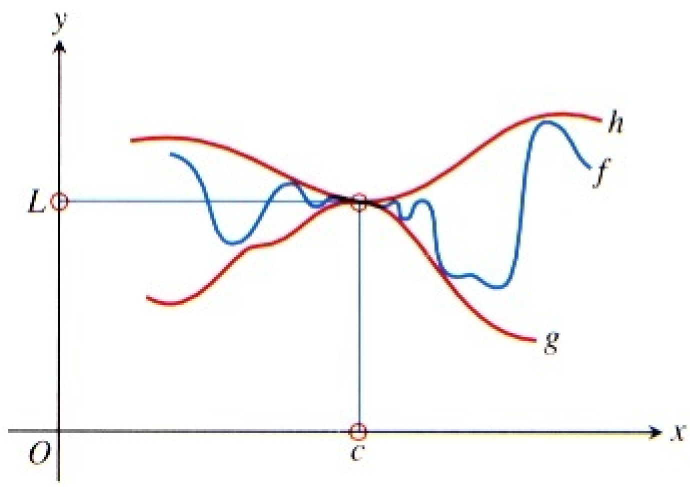

Figure 2.12

Note. Once we have a formal definition of limit, we can prove The Sandwich Theorem (see Appendix A.4). We illustrate the Sandwich Theorem (Theorem 2.4), also sometimes called "The Squeeze Theorem," in the next examples.

Examples. Example 2.2.11(a)(b) and Exercise 2.2.66(a).

Example. Exercise 2.2.79.