Chapter 2. Limits and Continuity

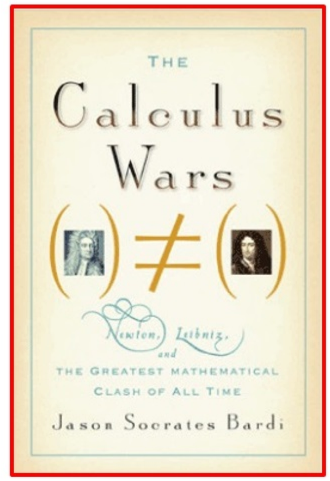
2.3 The Precise Definition of a Limit

Note. In this section, we give a mathematically rigorous definition of the limit of a function. We'll comment how the informal ideas from the previous section are justified by this definition. The concept of a limit is the idea behind all of calculus!!! You can perform the mechanical manipulations for most of the problems of this class without a deep understanding of limits (but so can software, including various websites such as Wolfram Alpha), but you cannot *really* understand the material of this class (including applications) without some level of understanding of the concept of a limit. So please invest some time in this section! But a warning: this is a tricky concept that took the mathematical community around 100 years to develop...

Note. Isaac Newton (December 25, 1642/January 4, 1643 – March 31, 1727) in 1666 wrote a tract that included many of the ideas of this course, including the Fundamental Theorem of Calculus (which we'll see in Section 5.4). However, his work was not published until after 1700. Gottfried Wilhelm Leibniz (July 1, 1646 – November 14, 1716) published two papers on calculus in 1684 and 1686. This resulted in a long argument between Newton and Leibniz over who was the first to invent or discover calculus (for details on this argument, see Jason Socrates Bardi's *The Calculus Wars: Newton, Leibniz, and the Greatest Mathematical Clash of All Time*, Basic Books, 2007). A simple version of the history of the genesis of calculus is that its properties were first studied by Newton and first published by Leibniz (this is oversimplified; Newton did not work in a vacuum and several others were involved in calculus-type ideas in the 17th century, and much earlier with Archimedes around 250 BCE).

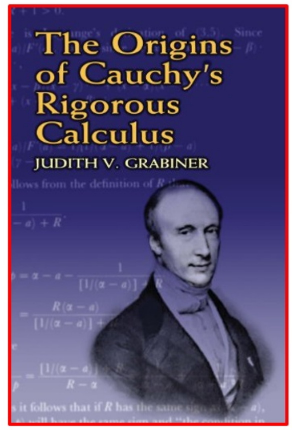
Isaac Newton

Gottfried W. Leibniz



Note. Archimedes, Newton, Leibniz and the rest did not have our formal definition of the limit of a function (in fact, our modern concept of a function appears after the work of Newton and Leibniz). Our definition is due to the French mathematician Augustin Louis Cauchy (August 21, 1789 – May 23, 1857). It is Cauchy that makes the informal ideas of “arbitrarily close” and “sufficiently close” formal in the early 1800s. Calculus grew from a somewhat informal endeavor to a mathematically rigorous area of study in the 1800s, and largely due to Augustin Cauchy. His contributions are spelled out in Judith V. Grabiner’s *The Origins of Cauchy’s Rigorous Calculus*, MIT Press, 1981 (today, this book is in print by Dover Publications).

Augustin Cauchy

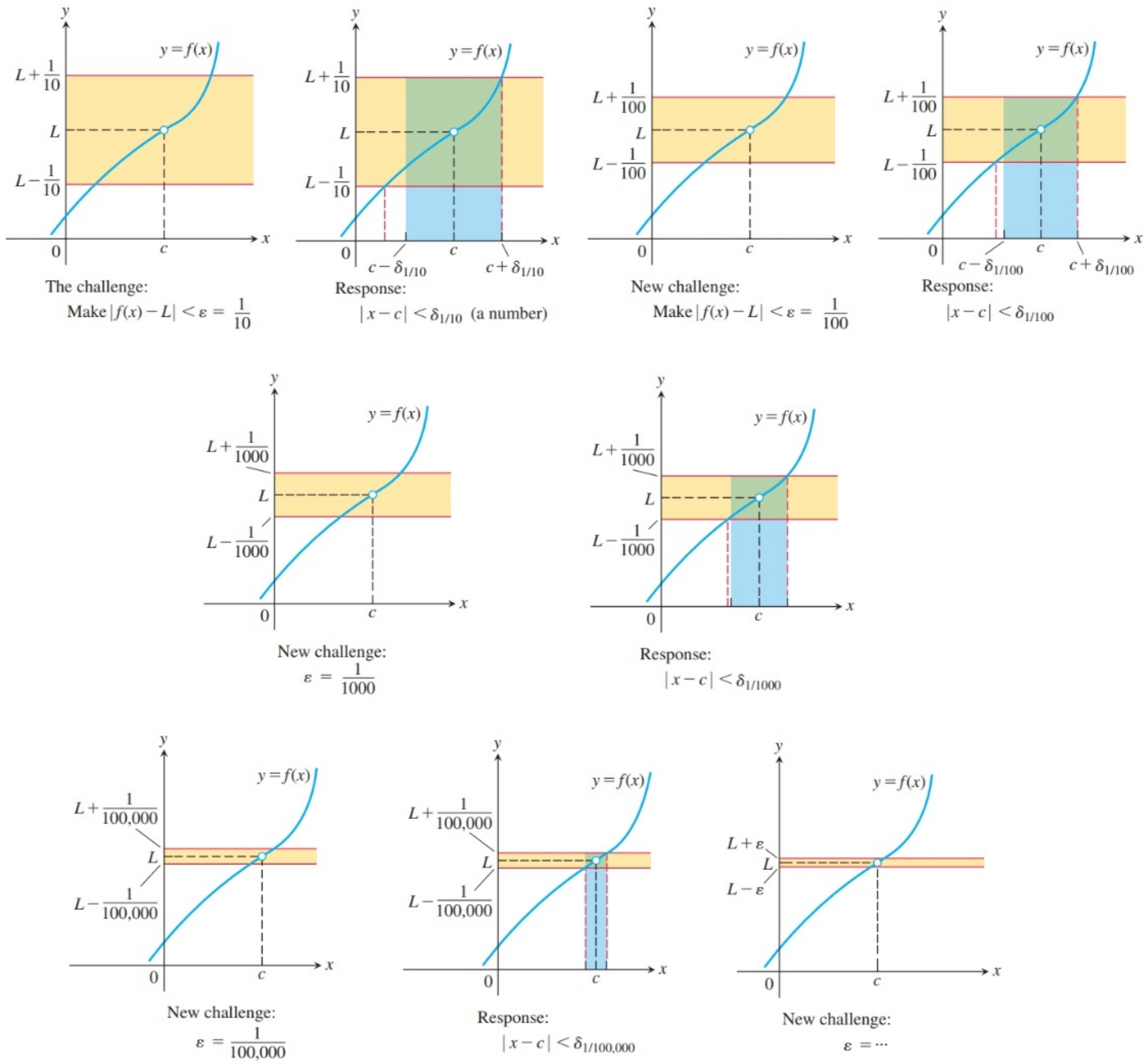


(The Newton, Leibniz, and Cauchy pictures are from MacTutor History of Mathematics Archive.) This Calculus 1 class is about an introduction to the ideas and manipulations of calculus, with an exposure to rigor but not an emphasis on rigor. The rigorous approach to limits, derivatives, integrals, sequences, and series are studied in ETSU's senior-level classes Analysis 1 (MATH 4217/5217) and Analysis 2 (MATH 4227/5227). I have notes posted online for both Analysis 1 and Analysis 2.

Example. As the book does, we introduce the formal definition of limit with an example. We consider Example 2.3.1.

Note. In the previous example, we wanted to make the function values within a distance of 2 of a certain value. In the formal definition of limit, we will want to make this distance arbitrarily small. In the following figure (from page 76 of the book) we consider a function that gets close to the value L when input value x is close to c. The figure shows, for distances between $y = f(x)$ and L of 1/10, 1/100, 1/1000, $1/100,000$, and ε (the last case being issued as a "challenge"), how close

x must be to c in order for $|y - L| = |f(x) - L|$ to be less than the given distance. The corresponding distances between x and c (which we measure as $|x - c|$) are denoted $\delta_{1/10}$, $\delta_{1/100}$, $\delta_{1/1000}$, and $\delta_{1/100{,}000}$, respectively.



With the given desired distance between $f(x)$ and L, denoted ε (the Greek letter epsilon), we want to find a corresponding distance between x and c, denoted δ (the Greek letter delta), such that when $|x - c| < \delta$ we have $|f(x) - L| < \varepsilon$. Geometrically, this means that the graph of $y = f(x)$ intersects the vertical sides of the little green box (where the horizontal and vertical bands intersect) and does not intersect the horizontal sides of the little green box (except possibly at the corners). Notice that none of the δ's shown above can be made any larger, since larger δ values will violate this intersection requirement between $y = f(x)$ and the little green boxes. As ε is made smaller, the δ likely must be made smaller, and the little green box gets smaller. But, as long as the graph of $y = f(x)$ "tries" to pass through the point (c, L) (and so $\lim_{x\to c} f(x) = L$), the little green box will close in around the point (c, L). This is why "Dr. Bobs Anthropomorphic Definition of Limit" is equivalent to what we are about to state.

Definition. Formal Definition of Limit

Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that $f(x)$ approaches the *limit* L *as* x *approaches* c and write $\lim\limits_{x\to c} f(x) = L$, if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - c| < \delta \text{ implies } |f(x) - L| < \varepsilon.$$

Note. We can now make the Informal Definition of Limit of the previous section more formal and clear. We had informally claimed that $\lim_{x\to c} f(x) = L$ if $f(x)$ gets "arbitrarily close" to L for all x "sufficiently close" to c (but not equal to c). Since $|f(x) - L|$ is the distance between $f(x)$ and L (and so reflects how close $f(x)$ is to L), and $|x - c|$ is the distance between x and c, then $\varepsilon > 0$ reflects the arbitrarily close idea and $\delta > 0$ reflects the sufficiently close idea. Since $\varepsilon > 0$ can be *anything*, then there is an arbitrariness to this distance. The value of δ

then depends on the choice of ε, as in the figure above. Once $\varepsilon > 0$ is given, the "challenge" is to find $\delta > 0$ such that *when* $0 < |x - c| < \delta$ *then* $|f(x) - L| < \varepsilon$. This means that when the x values are in the blue band, then the function values lie in the yellow band (and this is why the intersection of the graph of $y = f(x)$ must intersect the little green box as described above).

Note. The text book gives an illustration of the formal definition of limit in Figure 2.17. This is correct, but I prefer the following alternative illustration (which includes the yellow and blue bands, and the little green box):

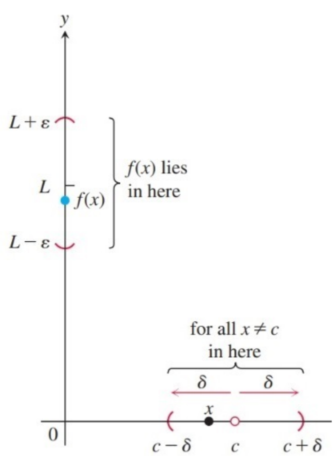
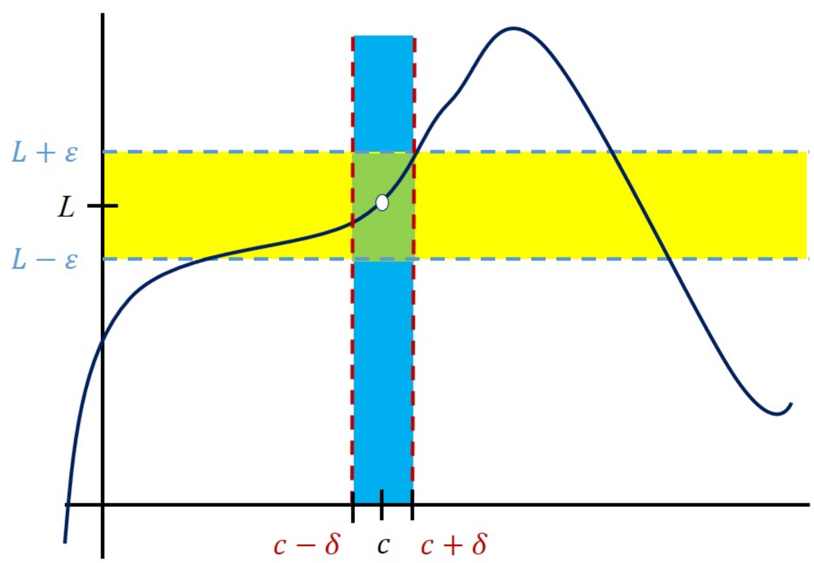


Figure 2.17



Notice that, in the formal definition of $\lim_{x \to c} f(x) = L$, we do not require that f is defined at c and that we only are concerned with x values satisfying $0 < |x - c| < \delta$; in particular, we avoid the value $x = c$. This is consistent with our observations from the previous section that it does not matter what happens at $x = c$! What matters, is the values of $f(x)$ for x near (and not equal to) c. The ε and δ quantities are what defines this nearness and "close to."

Note. You may hear limits described in terms of function values getting "closer and closer" to a limit value (you may even hear that the function "never gets there"). This is a common way try to convey the complicated idea of a limit, but it is not correct! It is not about getting closer and closer; if anything, it is about "getting close and staying close" (namely, $f(x)$ gets within $\varepsilon > 0$ of L and stays there for all $0 < |x-c| < \delta$). Since we require $|f(x)-L| < \varepsilon$, there is no prohibition of $f(x)$ taking on the value L (so f can "get there"); there is a prohibition against x taking on the value c in the requirement that $0 < |x - c|$, which implies $x \neq c$.

Note. We now illustrate the logic of the formal definition with a proof.

Example 2.3.A. Prove for $f(x) = mx + b$, $m \neq 0$, that $\lim\limits_{x \to a} f(x) = f(a)$.

Note. Example 2.3.A concerns a function whose graph is a line of slope $m \neq 0$. The text book gives a special case of this in Example 2.3.2 where a line of slope $m = 5$ is considered. We now give another example (the first part of which is also covered by Example 2.3.A).

Example. Example 2.3.3. Use the formal definition of limit to prove:
(a) $\lim_{x \to c} x = c$, **(b)** $\lim_{x \to c} k = k$ where k is a constant.

Note. Dealing with finding δ values for a given ε value is more difficult when the function has a graph that is not a line. We illustrate this first with an example.

Example. Example 2.3.4. For the limit $\lim_{x\to 5}\sqrt{x-1} = 2$ (true by the Root Rule, Theorem 2.1(7)), find $\delta > 0$ that works for $\varepsilon = 1$. That is, find a $\delta > 0$ such that

$$0 < |x-5| < \delta \text{ implies } |\sqrt{x-1} - 2| < 1.$$

Note. The previous example shows how we can use the graph of a function to find a value of δ that corresponds to a given ε value. However, graphing a function is very challenging (in fact, it is one of the things to which we apply calculus in Sections 4.3 and 4.4). So we need an algebraic way to find values for δ. For a function f which has a limit L at c, for a given $\varepsilon > 0$ we find a $\delta > 0$ such that

$$0 < |x-c| < \delta \text{ implies } |f(x) - L| < \varepsilon,$$

or equivalently (as the book states it)

$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x-c| < \delta,$$

as follows.

1. Solve the inequality $|f(x) - L| < \varepsilon$ to find an open interval (a, b) containing c on which the inequality holds for all $x \neq c$ (it doesn't matter what happens at $x = c$!).

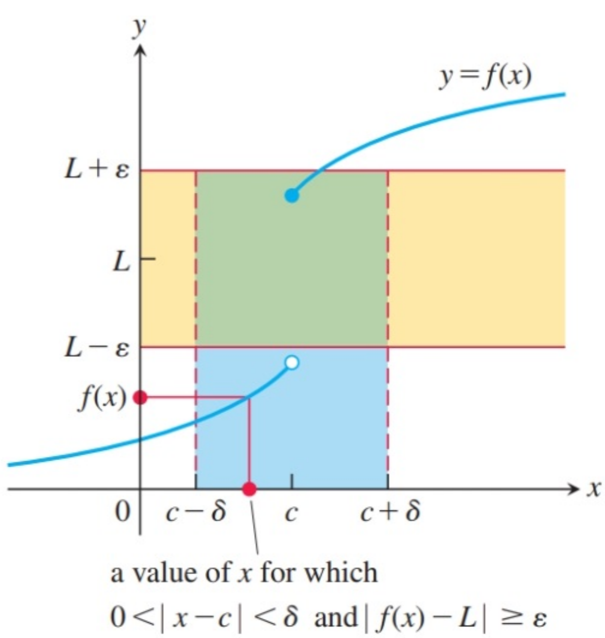
2. Find a value of $\delta > 0$ that places the open interval $(c-\delta, c+\delta)$ centered at c inside the interval (a, b). The inequality $|f(x) - L| < \varepsilon$ will hold for all $x \neq c$ in this δ-interval.

Examples. Exercise 2.3.20, and Exercise 2.3.40 (this one is complicated!).

Example 2.3.6. Prove the Sum Rule, Theorem 2.1(1): If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then

$$\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c}(f(x)) + \lim_{x\to c}(g(x)) = L + M.$$

Note. If we negate the formal definition of limit, we see that we can show that $\lim_{x\to c} f(x) \neq L$ if there is *some* $\varepsilon > 0$ such that *for all* $\delta > 0$ *there exists* some x value satisfying $0 < |x - c| < \delta$ and $|f(x) - L| \geq \varepsilon$. This means that the $\varepsilon > 0$ value is "bad" in the sense that no matter how we choose $\delta > 0$ there are still some x values that violate the required distance conditions. In the figure below we have chosen a bad $\varepsilon > 0$ such that no matter what $\delta > 0$ is, there is a bad x value (given in red).



Example. Exercise 2.3.58. Use the comment above to show that **(a)** $\lim_{x\to 2} h(x) \neq 4$, **(b)** $\lim_{x\to 2} h(x) \neq 3$, **(c)** $\lim_{x\to 2} h(x) \neq 2$ for the piecewise defined function

$$h(x) = \begin{cases} x^2, & x < 2 \\ 3, & x = 2 \\ 2, & x > 2. \end{cases}$$

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