Chapter 2. Limits and Continuity

2.5 Continuity

Note. We saw in Section 2.2 that, by Dr. Bob's Anthropomorphic Definition of Limit, a function f defined on an open interval about c, except possibly at c itself, has a limit L, $\lim_{x \to c} f(x) = L$, if the graph of $y = f(x)$ "tries" to pass through the point (c, L). The limit is unaffected by the behavior of f *at* $x = c$ (whether the function in fact succeeds in passing through the point or fails to pass through the point because either $f(c)$ is undefined or $f(c) \neq L$). We now consider an idea different from the existence of the limit, namely the idea of continuity. Informally, f is continuous at a point c if f tries to pass through the point $(c, f(c))$ and it succeeds! Since we have a formal definition of limit, we have the needed background to give a mathematically rigorous definition of continuity. We treat interior points c of the domain of f (where f is defined on some open interval containing c), left endpoints a of the domain (where f is defined on some interval $[a, d)$), and right endpoints b of the domain (where f is defined on some interval $(d, b]$) differently; we consider one-sided limits at endpoints of the domain.

Definition. Continuity at a Point.

Interior Point: A function $y = f(x)$ is *continuous at an interior point* c of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is *continuous at a left endpoint* a (or *continuous* *from the right* at a) or is *continuous at a right endpoint* b (or *continuous from the left* at b) of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively.}$$

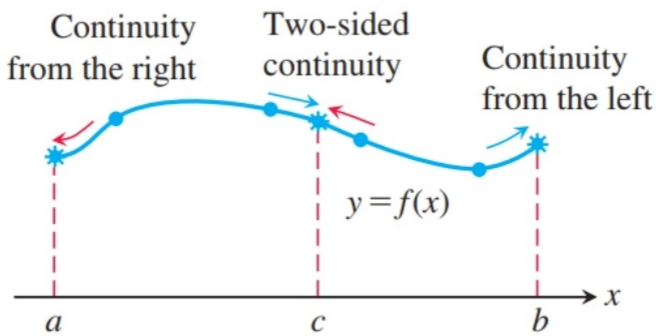


Figure 2.36

Note. We may also use the terminology "continuous from the left/right" at interior points of the domain. We see from Figure 2.36 that the idea of "trying to pass through a point" and succeeding holds for each of the three types of continuity. If a function is continuous at all points of its domain and the domain is an interval, we can also informally say that the function can be "drawn without picking up your pencil." We give an example involving the graph of a function and analyze it "anthropomorphically," before stating the Continuity Test.

Example. Exercise 2.5.4.

Note. In Section 2.2, we often dealt with limits by factoring, canceling, and substituting ("FCS"). A reasonable questions is: "When can we evaluate limits by substitution?" If there was an easy answer, we wouldn't have needed to go through all that ε and δ stuff! We can now, in hindsight, observe that a limit (one-sided or two-sided) can be evaluated by substitution when a function is *continuous* at the limit point (applying continuity at an endpoint when dealing with one-sided limits)! Why didn't we just say that before? Because the concept of continuity is **defined** in terms of limits!!! We can't use continuity to define limits, and then use limits to define continuity (no circular reasoning allowed). Also notice that a function is continuous if and only if we can pass limits in and out of the function: $\lim_{x\to c} f(x) = f\left(\lim_{x\to c} x\right)$. Similarly, we can pass an appropriate one-sided limit in and out of a function which is continuous at an endpoint of its domain.

Note. We can use the limit properties of polynomials (see Theorem 2.2) and rational functions (see Theorem 2.3) to show that they are continuous on their domains. The six trigonometric functions are also continuous, as will be established below. We then have the following.

Theorem 2.5.A. Polynomials, rational functions, and the six trigonometric functions are continuous at every point of their domains.

Note. We now break the definition of continuity into a three part checklist of requirements for continuity. As with the definition, we treat interior points of the domain and endpoints of the domain differently.

Note. We have the following test for continuity at (a) an interior point, (b) a left-hand endpoint, and (c) a right-hand endpoint of the domain of function f.

The Continuity Test.

(a) A function $f(x)$ is continuous at $x = c$, an interior point of the domain of f, if and only if it meets the following three conditions:

1. $f(c)$ exists,

2. $\lim_{x \to c} f(x)$ exists, and

3. $\lim_{x \to c} f(x) = f(c)$.

(b) A function $f(x)$ is continuous at $x = a$, a left-hand endpoint of the domain of f, if and only if it meets the following three conditions:

1. $f(a)$ exists,

2. $\lim_{x \to a^+} f(x)$ exists, and

3. $\lim_{x \to a^+} f(x) = f(a)$.

(c) A function $f(x)$ is continuous at $x = b$, a right-hand endpoint of the domain of f, if and only if it meets the following three conditions:

1. $f(b)$ exists,

2. $\lim_{x \to b^-} f(x)$ exists, and

3. $\lim_{x \to b^-} f(x) = f(b)$.

Example 2.5.A. Is this piecewise defined function continuous at $x = 0$:

$$f(x) = \begin{cases} x & \text{if } x \in (-\infty, 0) \\ 0 & \text{if } x = 0 \\ x^2 & \text{if } x \in (0, \infty)? \end{cases}$$

Note. We now define several different types of discontinuities (that is, points where a function is not continuous).

Definition. A function f has a *removable discontinuity* at $x = a$ if $f(a)$ can be redefined in such a way that f is continuous at a. Function f has a *jump discontinuity* at $x = a$ if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist (as finite numbers) and are different. Function f has an *infinite discontinuity* at $x = a$ if it is defined on an open interval containing a, except possibly at a itself, and it has a vertical asymptote at $x = a$ (we will formally define asymptotes in terms of limits in the next section).

Note. Some texts define removable discontinuities and jump discontinuities, and simply lump all other types of discontinuities into a single class of "a discontinuity of the third type." See see my online notes for Analysis 1 (MATH 4127/5127) on 4.2. Monotone and Inverse Functions.

Note. The book also considers *oscillating discontinuities*. We simply give a graph of a function with an oscillating discontinuity and do not formally define this idea. See Figure 2.40.

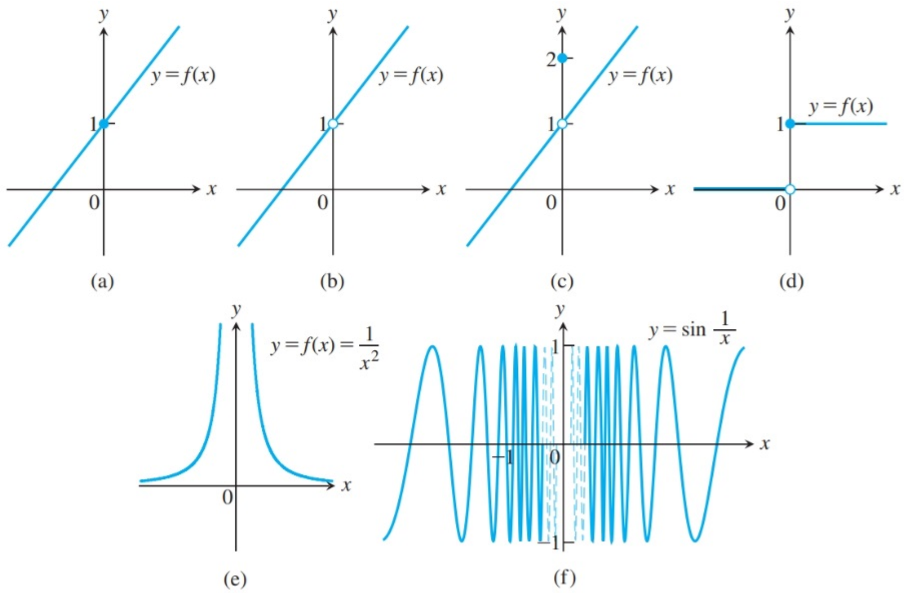


Figure 2.40. (a) A continuous function, (b) and (c) a function with a removable discontinuity at $a = 0$, (d) a function with a jump discontinuity at $a = 0$, (e) a function with an infinite discontinuity at $a = 0$, and (f) a function with an oscillating discontinuity at $a = 0$.

Note. If a function has a removable discontinuity at a point, then we can redefine the function at that point in such a way as to create a new function which *is* continuous at that point. This new function is called a *continuous extension* of the original function.

Examples. Discuss the discontinuities of $g(x) = \text{int } x$ (this is Example 2.5.4) and $f(x) = \dfrac{|x|}{x}$. Exercise 2.5.42.

Note. We now state a theorem, the proof of which is based on the corresponding limit results given in Theorem 2.1, "Limit Rules."

Theorem 2.8. Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums*: $f + g$

2. *Differences*: $f - g$

3. *Products*: fg

4. *Constant Multiples*: kf, for any number k

5. *Quotients*: f/g, provided $g(c) \neq 0$.

6. *Powers*: f^n, for a positive integer n

7. *Roots*: $\sqrt[n]{f}$, provided $\sqrt[n]{f}$ is defined on an open interval containing c, where n is a positive integer.

Example. Exercise 2.5.72: Prove that both $f(x) = \sin x$ and $g(x) = \cos x$ are continuous at every point $x = c$.

Note. Since each of the six trigonometric functions can be written as products or quotients of $\sin x$ and $\cos x$, then we can use Theorem 2.8(5) and Exercise 2.5.72 to show that all six trig functions are continuous on their domains. This completes the argument for Theorem 2.5.A.

Note. If a function f is continuous on an interval and if f has an inverse, f^{-1}, then the inverse function is continuous. This is established in "more advanced texts"; see my online notes for Analysis 1 (MATH 4127/5127) on 4.2. Monotone and Inverse Functions (see Theorem 4-16. Continuity of the Inverse Function.). We accept this and so have that inverse trigonometric functions are continuous on their domains (this requires us to deal with some endpoints of the domains for $\text{arcsec}\, x$ and $\text{arccsc}\, x$; see Figure 1.64). In Chapter 7, we will show that the exponential function $f(x) = a^x$, where $a > 0$ and $a \neq 1$, is continuous. Therefore its inverse function, $f^{-1}(x) = \log_a x$ is continuous on its domain. In particular, we have the natural exponential function $f(x) = e^x$ and the natural logarithm function $f^{-1}(x) = \ln x$ are continuous on their domains. So we take the following to be true.

Theorem 2.5.B. The six inverse trigonometric functions, exponential functions a^x where $a > 0$ and $a \neq 1$, and logarithm functions $\log_a x$ where $a > 0$ and $a \neq 1$, are continuous on their domains.

Note. We now turn our attention to limits of compositions of continuous functions. A proof of the next result is to be given in Exercise A.4.6.

Theorem 2.9. Compositions of Continuous Functions

If f is continuous at c and g is continuous at $f(c)$, then the composition $g \circ f$ is continuous at c.

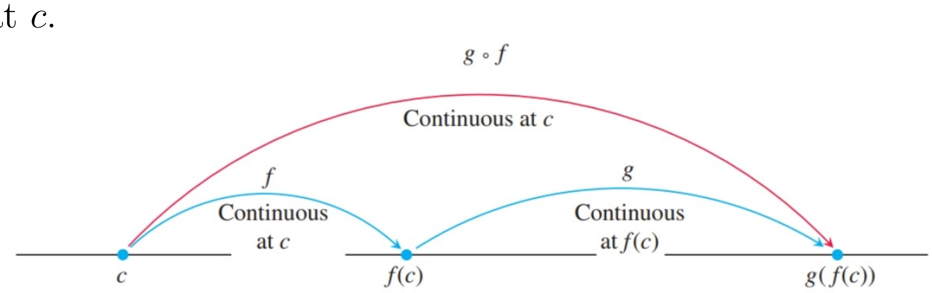


Figure 2.42

Example. Exercise 2.5.26.

Note. Theorem 2.9 is a special case of the more following result. We give a rigorous proof.

Theorem 2.10. Limits of Continuous Functions.
If g is continuous at the point b and $\lim_{x \to c} f(x) = b$, then

$$\lim_{x \to c} g(f(x)) = g(b) = g(\lim_{x \to c} f(x)).$$

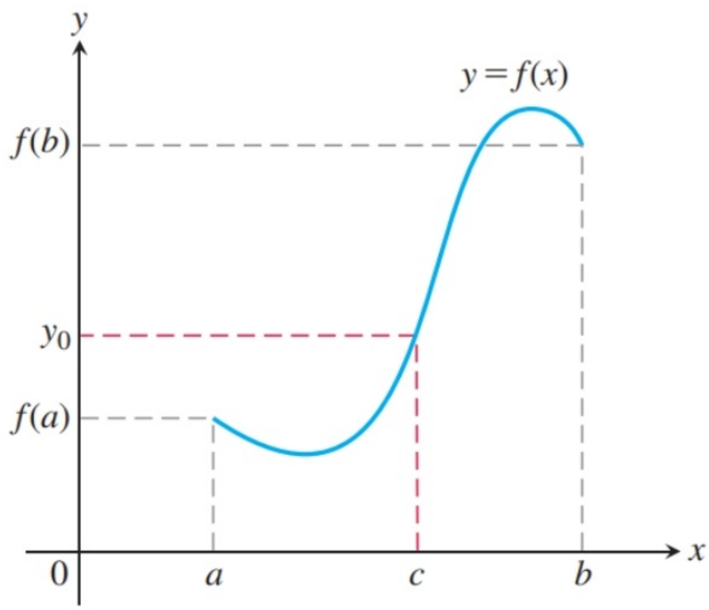
Note. In Theorem 2.10, when considering $\lim_{x \to c} f(x)$, we assume that f is defined on an open interval containing c, except possibly at c itself. Theorem 2.10 also holds for endpoints of the domain if we replace the limit with an appropriate one-sided limit.

Example. Exercise 2.5.34.

Note. The next result is fundamental! For a proof, see my online notes for Analysis 1 (MATH 4127/5127) on 4.1. Limits and Continuity (see Corollary 4-9). This requires a rather deep understanding of the real numbers.

Theorem 2.11. The Intermediate Value Theorem for Continuous Functions

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



Note. Notice the intuitive nature of the Intermediate Value Theorem: Think of it as saying that you can't get from one side of a road to the other side without crossing the center line (or any line painted on the road in the direction of the road). We can use the Intermediate Value Theorem to show the existence of certain x values without actually finding the x values. This is illustrated in the next example.

Example. Exercise 2.5.56.

Example. Exercise 2.5.68.

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