Chapter 2. Limits and Continuity

2.6 Limits Involving Infinity; Asymptotes of Graphs

Note. In this section we give a calculus meaning to the symbol ∞. So far, you have probably only seen this symbol in connection with unbounded intervals: $\mathbb{R} = (-\infty, \infty)$, $(-\infty, 0] = \{x \in \mathbb{R} \mid x \leq 0\}$, or $(7, \infty) = \{x \in \mathbb{R} \mid x > 7\}$. As the text book comments, ∞ is not a number! You will *never* do arithmetic with the ∞ symbol. In a calculus class, ∞ is a limit. Since you are now familiar with formal definitions of limits, we start there. We will also give an informal definition, but won't have an anthropomorphic definition this time (in part, because there is no such thing as "close to ∞").

Definition. Formal Definition of Limits at Infinity.

1. Let f be a function such that for some real number P, the domain of f includes (P, ∞). We say that $f(x)$ has the *limit L as x approaches infinity* and we write $\lim_{x\to\infty} f(x) = L$ if, for every number $\varepsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \text{ implies } |f(x) - L| < \varepsilon.$$

2. Let f be a function such that for some real number P, the domain of f includes $(-\infty, P)$. We say that $f(x)$ has the *limit L as x approaches negative infinity* and we write $\lim_{x\to-\infty} f(x) = L$ if, for every number $\varepsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \text{ implies } |f(x) - L| < \varepsilon.$$

Definition. Informal Definition of Limits Involving Infinity.

1. Let f be a function such that for some real number P, the domain of f includes (P, ∞). We say that $f(x)$ has the *limit L as x approaches infinity* and write $\lim_{x\to\infty} f(x) = L$ if $f(x)$ gets arbitrarily close to L as x moves sufficiently far from the origin in the positive direction.

2. Let f be a function such that for some real number P, the domain of f includes $(-\infty, P)$. We say that $f(x)$ has the *limit L as x approaches negative infinity* and write $\lim_{x\to-\infty} f(x) = L$ if $f(x)$ gets arbitrarily close to L as x moves sufficiently far from the origin in the negative direction.

Note. Notice that when we write $x \to \infty$ (or $x \to -\infty$) we do not mean that x is close to infinity (there is no ∞ point on the real number line). Instead, we refer to x as "moving sufficiently far from the origin in the positive direction" (and analogously for $x \to -\infty$). We now give a proof of a result which will be fundamental in our computations of limits as $x \to \pm\infty$.

Example. Example 2.6.1(a). Prove that $\lim_{x\to\infty} \frac{1}{x} = 0$.

Note. Similar to Example 2.6.1(a), we can show that $\lim_{x\to-\infty} \frac{1}{x} = 0$. We simply choose $N = -1/\varepsilon$ in the proof (as is shown in Example 2.6.1(b).

Note. Each of the claims in Theorem 2.1 (Limit Rule) also holds as $x \to \pm\infty$, as follows.

Theorem 2.12. Rules for Limits as $x \to \pm\infty$.

If L, M, and k are real numbers and

$$\lim_{x\to\pm\infty} f(x) = L \text{ and } \lim_{x\to\pm\infty} g(x) = M, \text{ then}$$

1. *Sum Rule*: $\lim_{x\to\pm\infty}(f(x)+g(x)) = \lim_{x\to\pm\infty} f(x) + \lim_{x\to\pm\infty} g(x) = L + M$

2. *Difference Rule*: $\lim_{x\to\pm\infty}(f(x)-g(x)) = \lim_{x\to\pm\infty} f(x) - \lim_{x\to\pm\infty} g(x) = L - M$

3. *Product Rule*: $\lim_{x\to\pm\infty}(f(x)g(x)) = \left(\lim_{x\to\pm\infty} f(x)\right)\left(\lim_{x\to\pm\infty} g(x)\right) = LM$

4. *Constant Multiple Rule*: $\lim_{x\to\pm\infty}(kf(x)) = k\lim_{x\to\pm\infty} f(x) = kL$

5. *Quotient Rule*: $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x\to\pm\infty} f(x)}{\lim_{x\to\pm\infty} g(x)} = \frac{L}{M}, M \neq 0$

6. *Power Rule*: If n is a positive integer, then $\lim_{x\to\pm\infty}(f(x))^n = \left(\lim_{x\to\pm\infty} f(x)\right)^n = L^n$.

7. *Root Rule*: If n is a positive integer, then $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to\pm\infty} f(x)} = \sqrt[n]{L} = L^{1/n}$ (if n is even, we also require that $f(x) \geq 0$ on some interval of the form (P,∞) in the event that $x \to \infty$, or $f(x) \geq 0$ on some interval of the form $(-\infty, P)$ in the event that $x \to -\infty$).

Note. When dealing with limits as $x \to \pm\infty$ for rational functions, we first divide the numerator and denominator by the highest power of x in the denominator and then use Examples 2.6.1(a) and (b).

Examples. Exercise 2.6.14 and Exercise 2.6.36.

Note. We now take our first step (but not our last) in using calculus to graph a function.

Definition. Horizontal Asymptote.

A line $y = b$ is a *horizontal asymptote* of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \text{ or } \lim_{x\to-\infty} f(x) = b.$$

Example. Exercise 2.6.68, find the horizontal asymptotes.

Example. Example 2.6.4. This example shows that a function can have two horizontal asymptotes.

Example. Example 2.6.5: Prove $\displaystyle\lim_{x\to-\infty} e^x = 0$.

Note. Theorem 2.10, "Limits of Continuous Functions," holds for limits as $x \to \pm\infty$:

> **Theorem 2.6.A.** If g is continuous at the point b and $\displaystyle\lim_{x\to\pm\infty} f(x) = b$, then
>
> $$\lim_{x\to\pm\infty} g(f(x)) = g(b) = g\left(\lim_{x\to\pm\infty} f(x)\right).$$

Example 2.6.A. Evaluate $\lim_{x\to\infty} \cos(1/x)$.

Note. Theorem 2.4, "Sandwich Theorem," holds for limits as $x \to \pm\infty$:

> **Theorem 2.6.B.** Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval of the form (P, ∞). Suppose that $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} h(x) = L$. Then $\lim_{x\to\infty} f(x) = L$.
> Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval of the form $(-\infty, P)$. Suppose that $\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} h(x) = L$. Then $\lim_{x\to-\infty} f(x) = L$.

Example. Example 2.6.8.

Note. In Example 2.6.8, we see that the graph of $y = 2 + \dfrac{\sin x}{x}$ has a horizontal asymptote of $y = 2$. You may have hear it said before that an asymptote is something that the graph of the function gets "closer and closer" to "but never gets there." This is simply **wrong**! As you can see from the graph of $y = f(x) = 2 + (\sin x)/x$, f actually does "get to" it's limit value of 2; in fact, $f(x) = 2$ for every integer multiple of π except 0. In addition, the graph does not get closer to the limit value of 2. In fact, on all intervals of the form $(k\pi, k\pi + \pi/2)$, where k is a positive integer, the graph is getting further away from 2 (as x increases)! A similar observation holds for intervals of the form $(k\pi, k\pi - \pi/2)$ where k is a negative integer). See Figure 2.57 below. If you want a simple (correct) way to informally explain horizontal asymptotes, then say that the graph "gets close to and stays close to" the asymptote.

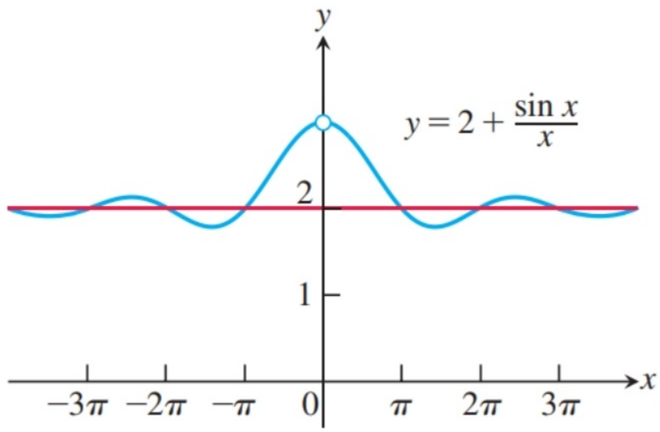


Figure 2.57

If a rational function g has a horizontal asymptote, then it is true that for "sufficiently large" values of x the graph of $y = g(x)$ gets closer and closer to the asymptote (as x gets larger) and does not "get there" (that is, take on the asymptotic value) for sufficiently large x. However, even in this case the g can intersect the horizontal asymptote. For other details, see my brief publication R. Gardner, Horizontal Asymptotes: What They are Not, *The Mathematics Teacher* (Reader Reflections), February 1998, 152.

Example. Exercise 2.6.92.

Definition/Note. If the degree of the numerator of a rational function is one greater than the degree of the denominator, the graph has an *oblique asymptote* (or *slant asymptote*). The asymptote is found by dividing the denominator into the numerator to express the function as a linear term plus a term that goes to zero as $x \to \pm\infty$.

Note. In Example 2.6.10, the oblique asymptote of $f(x) = \dfrac{x^2 - 3}{2x - 4}$ is shown to be $y = x/2 + 1$. The graph of $y = f(x)$ is given in Figure 2.58. Notice that f also has a vertical asymptote at $x = 2$ (we explore vertical asymptotes using limits next).

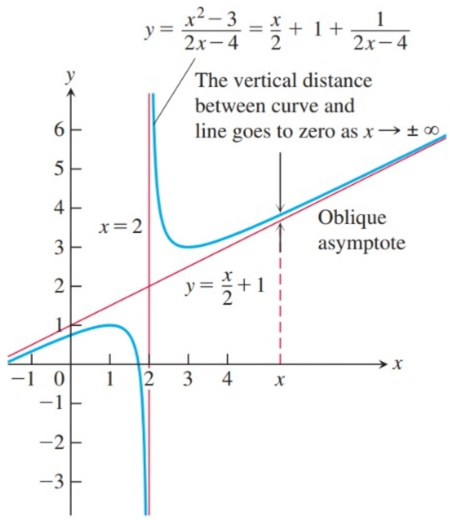


Figure 2.58

Example. Exercise 2.6.108, find the oblique asymptote.

Note. So far in this section, we have considered what happens when the *input* x of a function f approaches $\pm\infty$. We now consider the possibility that the *output* values $f(x)$ can get arbitrarily large. To motivate our approach consider the graph of $y = f(x) = 1/x$ given in Figure 2.59 below.

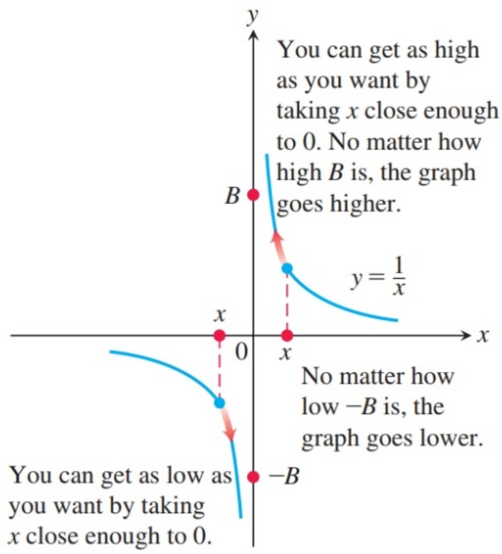


Figure 2.59

Consider what happens to the graph when $x \to 0^+$. As illustrated in Figure 2.59, no matter how big positive number B is, the graph of $y = 1/x$ can be made to lie above B by making x sufficiently close to 0 and positive. As for $x \to 0^-$, no matter how big negative number $-B$ is, the graph of $y = 1/x$ can be made to lie below $-B$ by making x sufficiently close to 0 and negative. This motivates us to introduce the notation $\lim_{x\to 0^-} 1/x = -\infty$ and $\lim_{x\to 0^+} 1/x = \infty$. Notice that it is meaningless to describe the function values as "getting close to" infinity (since there is no location of $\pm\infty$ on the real number line); instead, we describe the function values as getting arbitrarily large in the positive or negative direction. Notice from Figure 2.59 that the graph of $y = 1/x$ has a vertical asymptote at $x = 0$. We will use these types of limits to define the term "vertical asymptote." First, we formally define these types of infinite limits. We present the definition for two-sided limits and observe that these can be modified to address one sided infinite limits.

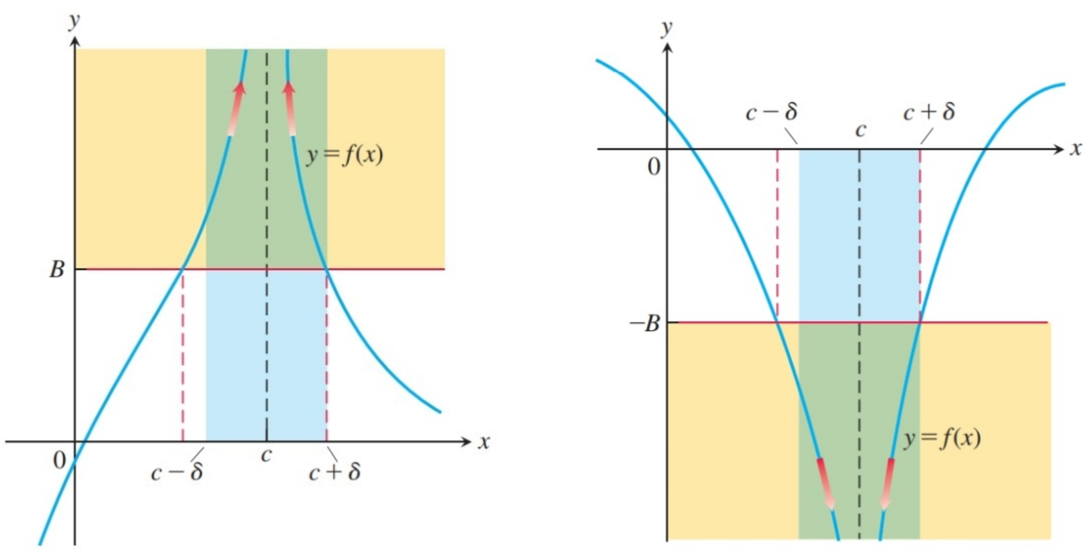
Definition. Infinity, Negative Infinity as Limits. Let f be a function defined on an open interval containing c, except possibly at c itself.

1. We say that $f(x)$ *approaches infinity as x approaches* c, and we write $\lim_{x\to c} f(x) = \infty$, if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - c| < \delta \text{ implies } f(x) > B.$$

2. We say that $f(x)$ *approaches negative infinity as x approaches* c, and we write $\lim_{x\to c} f(x) = -\infty$, if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - c| < \delta \text{ implies } f(x) < -B.$$



Figures 2.62 and 2.63

Note. Informally, $\lim\limits_{x\to c} f(x) = \infty$ if $f(x)$ can be made arbitrarily large by making x sufficiently close to c (and similarly for f approaching negative infinity). We can also define one-sided infinite limits in an analogous manner (see Exercise 2.6.99).

Example 2.6.B. For n a positive even integer, prove that $\lim_{x\to 0} \frac{1}{x^n} = \infty$.

Definition. A line $x = a$ is a *vertical asymptote* of the graph if either

$$\lim_{x\to a^+} f(x) = \pm\infty \text{ or } \lim_{x\to a^-} f(x) = \pm\infty.$$

Note. Recall that we look for the vertical asymptotes of a rational function where the denominator is zero (though just because the denominator is zero at a point, the function does not *necessarily* have a vertical asymptote at that point). We make things more precise in the following result:

> **Dr. Bob's Infinite Limits Theorem.** Let $f(x) = \frac{p(x)}{q(x)}$. Suppose $\lim_{x\to c} p(x) = L \neq 0$, $\lim_{x\to c} q(x) = 0$, and $q(x)$ is of the same sign on some open interval containing c, except possibly at c itself. Then $\lim_{x\to c} f(x) = \pm\infty$. We can say something similar for one-sided limits.

Note. We can simplify Dr. Bob's Infinite Limits Theorem by applying it to rational functions. It then becomes: "Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function. Suppose $\lim_{x\to c^+} p(x) = L \neq 0$ and $\lim_{x\to c^+} q(x) = 0$. Then $\lim_{x\to c^+} f(x) = \pm\infty$." We can say something similar for limits from the left.

Examples. Exercise 2.6.54 and Exercise 2.6.70.

Note. The last topic in this section is a generalization of the idea of an oblique asymptote. For a polynomial function $f(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_2 x^2 + c_1 x + c_0$, a *dominant term* as $x \to \pm\infty$ is the function $g(x) = c_n x^n$. The idea is that for x large, $f(x)$ and $g(x)$ are roughly the same. More precisely, $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = 1$. So for large x, the graph of $y = f(x)$ and the graph of $y = g(x)$ are close to each other. We illustrate this with a specific example.

Example. Example 2.6.20.

Examples. Exercise 2.6.108 (again), Exercise 2.6.80, Exercise 2.6.102.

Revised: 7/25/2020