Chapter 3. Derivatives

Note. In this chapter we apply limits to define the derivative of a function. The derivative is related to the instantaneous rate of change introduced in Section 2.1. We'll derive several rules of differentiation and apply derivatives to rate of change problems. This chapter is a bit more computational (and less theoretical) than Chapter 2.

3.1. Tangent Lines and the Derivative at a Point

Note. In this section we define the slope of a curve at a point. We take our inspiration from the slopes of secant lines and tangent lines to the function, which were introduced in Section 2.1.

Definition. The *slope of the curve* $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists. The *tangent line* to the curve at P is the line through P with this slope. See Figure 3.1 below.

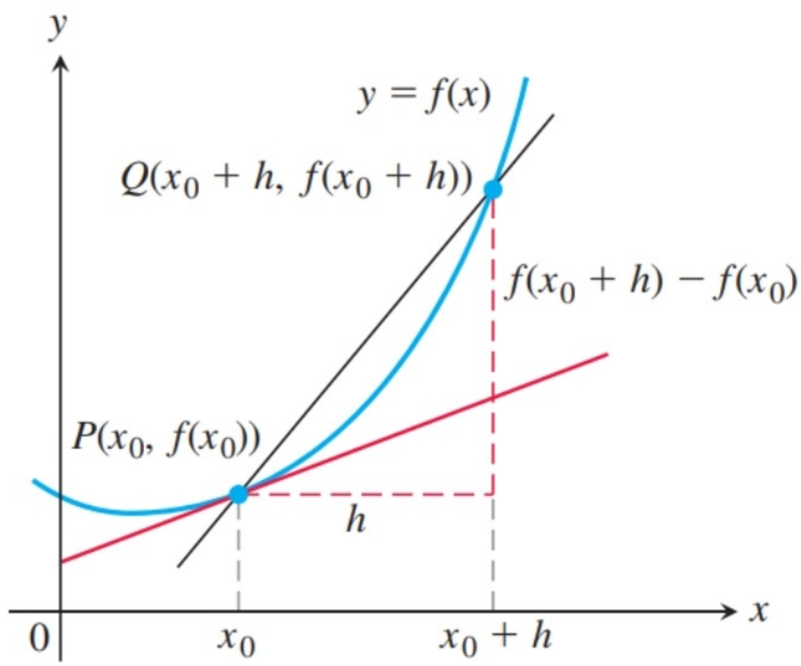


Figure 3.1

Examples. Exercise 3.1.7 and Exercise 3.1.12.

Definition. The expression $\dfrac{f(x_0+h)-f(x_0)}{h}$, where $h \neq 0$, is the *difference quotient of f at x_0 with increment h*. The *derivative of a function f at a point x_0*, denoted $f'(x_0)$, is the limit as $h \to 0$ of the difference quotient:

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h},$$

provided this limit exists.

Example. Exercise 3.1.28.

Note. Since the derivative of a function at a point is a limit of an average rate of change (to recall a topic from Section 2.1), then we see that the derivative can be interpreted as an instantaneous rate of change of the function f with respect to the variable x. For example, if $f(t)$ is the position of a particle at time t, then the instantaneous rate of change of position with respect to time (i.e. the *instantaneous velocity*) at time $t = t_0$ is

$$f'(t_0) = \lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h},$$

provided the limit exists.

Example. Exercise 3.1.30.

Note. We have several interpretations of the limit of the difference quotient

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$.

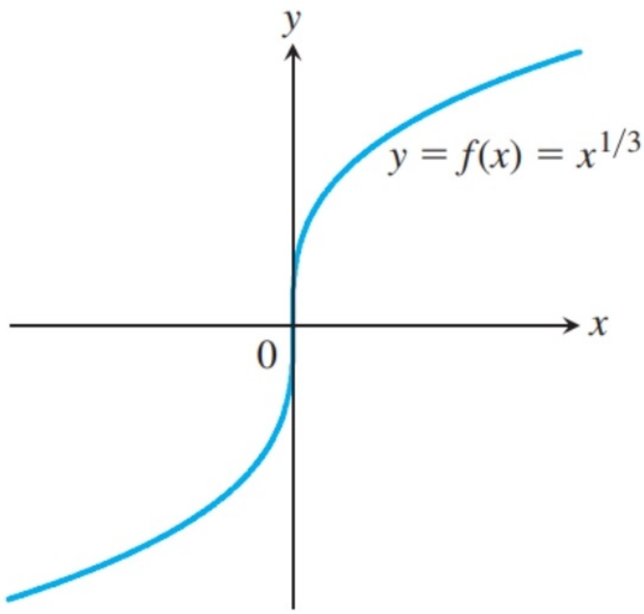
2. The slope of the tangent line to the curve $y = f(x)$ at $x = x_0$.

3. The rate of change of $f(x)$ with respect to x at $x = x_0$.

4. The derivative $f'(x_0)$ at $x = x_0$.

Definition. A continuous curve $y = f(x)$ has a *vertical tangent line* at the point where $x = x_0$ if the (two-sided) limit of the difference quotient is ∞ or $-\infty$.

Note. We can verify that $\lim_{h\to 0} \dfrac{f(0+h)-f(0)}{h} = \infty$ for $f(x) = x^{1/3}$. Therefore the graph $y = x^{1/3}$ has a vertical tangent line at $x = 0$:



VERTICAL TANGENT LINE AT ORIGIN

Example. Exercise 3.1.42.

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