# Chapter 3. Derivatives

## 3.11. Linearization and Differentials

**Note.** In this section we approximate functions (locally) with functions whose graphs are lines; these second functions are called "linearizations." Linearizations are based on tangent lines to a function. We will also finally give a proof of the Chain Rule (Theorem 3.2).

**Definition.** If $f$ is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the *linearization* of $f$ at $a$. The approximation $f(x) \approx L(x)$ of $f$ by $L$ is the *standard approximation* of $f$ at $a$. The point $x = a$ is the *center* of the approximation.

**Note.** Since $L(x)$ is the line tangent to the graph of $y = f(x)$ at $x = a$, then we expect $L(x)$ to be a good approximation of $f(x)$ when $x$ is close to $a$. See Figure 3.52.

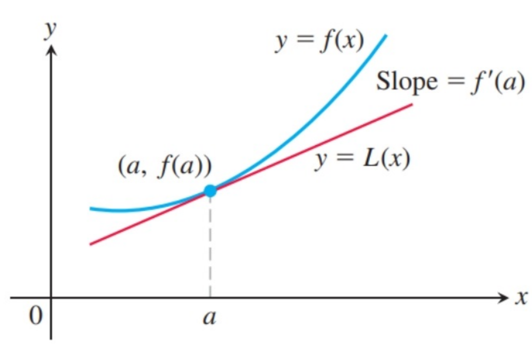


**Figure 3.52**

**Note.** The text book repeatedly pushes the idea that if we magnify our view of the graph of $y = f(x)$ near point $(a, f(a))$ then the graph of the linearization $y = L(x)$ closely approximates $y = f(x)$, with the greater magnification giving the better approximation. See Figures 3.53 and 3.54 where $f(x) = \sqrt{1+x}$ is linearized as $L(x) = 1 + x/2$ at $a = 0$ (this is Example 3.11.1).

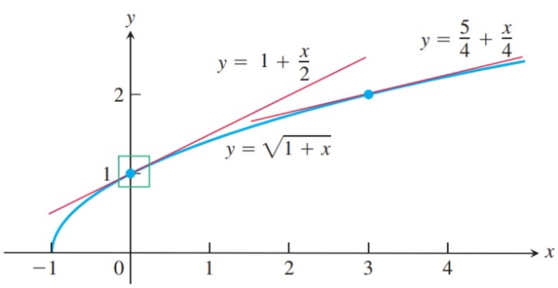


**FIGURE 3.53** The graph of $y = \sqrt{1+x}$ and its linearizations at $x = 0$ and $x = 3$. Figure 3.54 shows a magnified view of the small window about 1 on the $y$-axis.

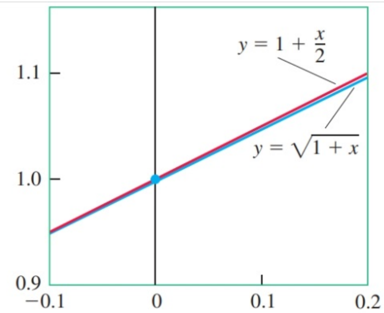


**FIGURE 3.54** Magnified view of the window in Figure 3.53.

**Example.** Exercise 3.11.2.

**Definition.** Let $y = f(x)$ be a differentiable function. The *differential* $dx$ is an independent variable. The *differential* $dy$ is

$$dy = f'(x)\,dx.$$

**Examples.** Exercise 3.11.28 and 3.11.38.

**Note.** Let $f(x)$ be differentiable at $x = a$. The approximate change (the differential estimate of change) in the value of $f$ when $x$ changes from $a$ to $a + dx$ is the change

$\Delta L$ in the linearization of $f$ at $a$,

$$\Delta L = df = f'(a)\,dx.$$

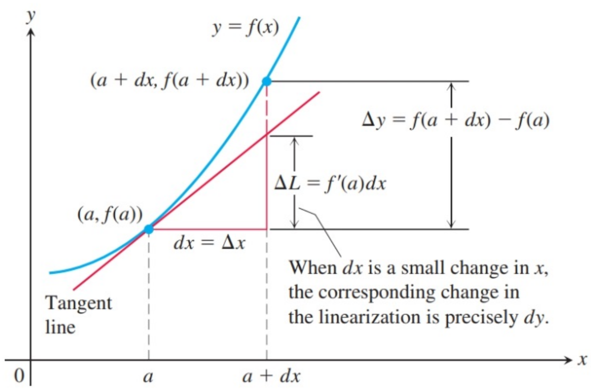


**Figure 3.56**

**Note.** If we know the value of a differentiable function $f(x)$ at a point $a$ and want to estimate the value of $f$ when the $x$ value changes from $a$ to $a + \Delta x$, then with $\Delta x = dx$ we have

$$f(a + \Delta x) = f(a + dx) = f(a) + \Delta y \approx f(a) + dy.$$

So the change in $f$ is $\Delta y \approx dy$ where $dy$ (a function of $x$ and $dx$) is evaluated at $a$ as $dy = f'(a)\,dx$.

**Example 3.11.A.** Use differentials to estimate the value of $\sin 31°$.

**Example.** Exercise 3.11.44.

**Definition.** We can compare actual changes in a function and the estimated change which is calculated from the use of differentials. We consider the absolute, relative, and percentage change:

| | **True** | **Estimated** |
|---|---|---|
| Absolute change | $\Delta f = f(a + dx) - f(a)$ | $df = f'(a)\,dx$ |
| Relative change | $\dfrac{\Delta f}{f(a)}$ | $\dfrac{df}{f(a)}$ |
| Percentage change | $\dfrac{\Delta f}{f(a)} \times 100\%$ | $\dfrac{df}{f(a)} \times 100\%$ |

**Examples.** Exercise 3.11.56 and Exercise 3.11.58.

**Note.** We now present a proof of the Chain Rule. First, we need a preliminary lemma.

**Lemma 3.11.A.** If $y = f(x)$ is differentiable at $x = a$ and $x$ changes from $a$ to $a + \Delta x$, the corresponding change $\Delta y$ in $f$ is given by $\Delta y = f'(a) + \varepsilon\Delta x$ in which $\varepsilon \to 0$ as $\Delta x \to 0$.

**Theorem 3.2. The Chain Rule.**

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at $x$, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at $x$, and

$$(f \circ g)'(x) = f'(g(x))[g'(x)].$$