Chapter 3. Derivatives

3.2. The Derivative as a Function

Note. In Section 3.1 we defined the derivative of a function f at a point x_0, $f'(x_0)$. In this section we define the derivative as a function itself and compute the derivatives of several functions. We still use limits to compute derivatives, but will speed up the process starting in the next section in which we state some rules of differentiation.

Definition. Derivative Function.

The *derivative* of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. Function f is *differentiable* at each value for which f' exists.

Note. Motivated by Sections 2.1 and 3.1, we see that $f'(x)$ is the slope of the line tangent to $y = f(x)$ as a function of x. There are a number of ways to denote the derivative of $y = f(x)$:

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}[f].$$

Examples. Exercise 3.2.10 and Example 3.2.3.

Note. From the graph of $y = f'(x)$ we can see:

1. where the rate of change of f is positive, negative, or zero;

2. the rough size of the growth rate of f at any x;

3. where the rate of change itself is increasing or decreasing.

Examples. Exercise 3.2.14, Exercise 3.2.24, and Exercise 3.2.30.

Definition. A function f is *differentiable on an open interval* if $f'(x)$ is defined for each x in the open interval. Function f is *differentiable on a closed interval* $[a, b]$ if it is differentiable on the interval (a, b) and if the limits:

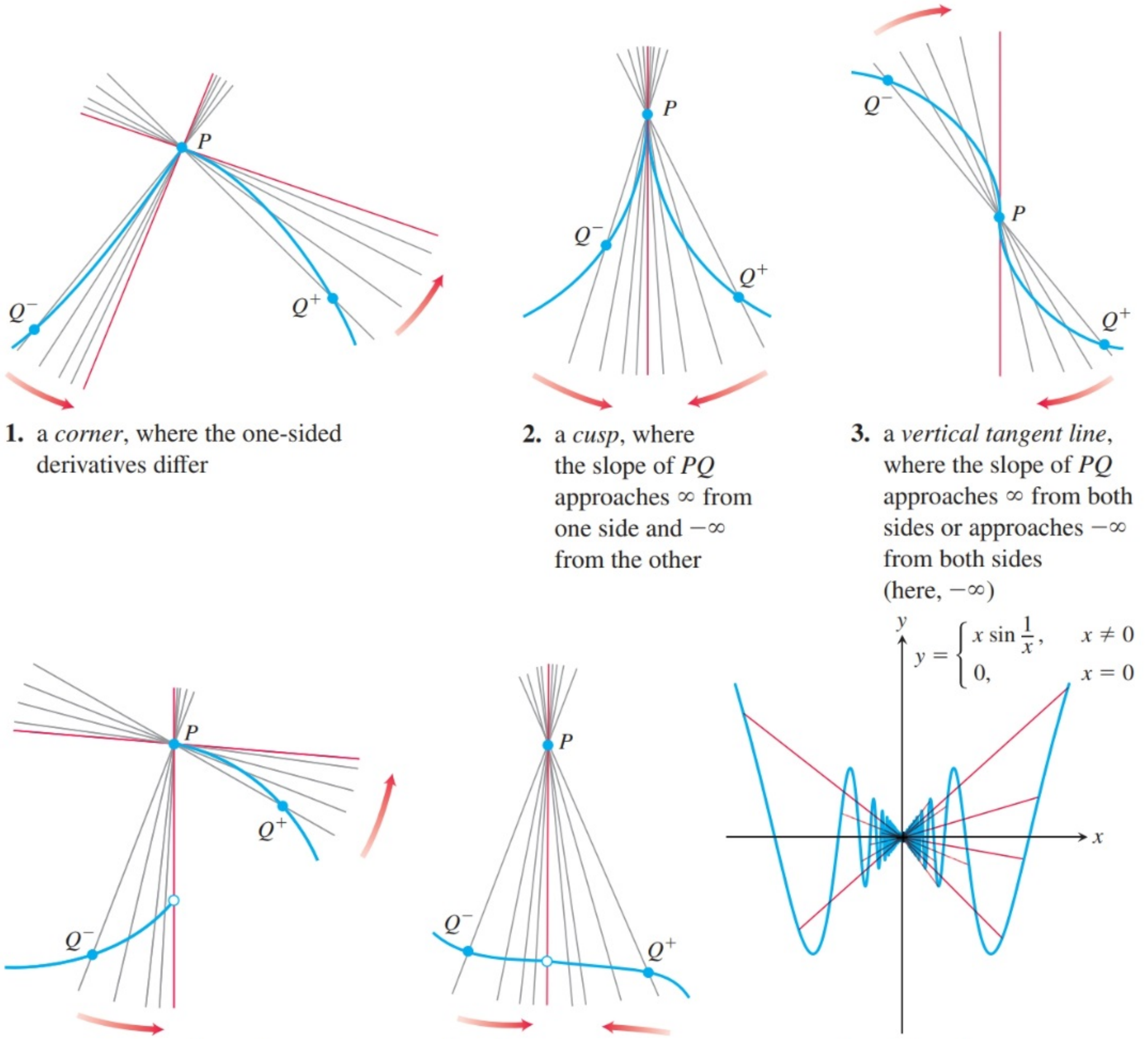
$$\text{Right-hand derivative at } a: \quad \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

$$\text{Left-hand derivative at } b: \quad \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$$

exist at the endpoints a and b.

Example. Exercise 3.2.44.

Note. There are a number of reasons as to why a function might not have a derivative at a point. Some of these reasons are illustrated here:



1. a *corner*, where the one-sided derivatives differ

2. a *cusp*, where the slope of *PQ* approaches ∞ from one side and $-\infty$ from the other

3. a *vertical tangent line*, where the slope of *PQ* approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$)

4. a *discontinuity* (two examples shown)

5. wild oscillation

Theorem 3.1. Differentiability Implies Continuity

If f has a derivative at $x = c$, then f is continuous at $x = c$.

Note. The converse of Theorem 3.1 does not hold. That is, a function can be continuous at a point and yet not differentiable. It is shown in Example 3.2.4 that the continuous function $f(x) = |x|$ is not differentiable at 0. In fact there is a continuous function that isn't differentiable anywhere. In Exercise 3.2.64 it is claimed that the Weierstrass function $f(x) = \displaystyle\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \cos(9^n \pi x)$ is continuous for all $x \in \mathbb{R}$ but the derivative doesn't exist for any $x \in \mathbb{R}$. (The function is presented as an infinite "series," a topic that is explored in Calculus 2.)

Note. Karl Wilhelm Weierstrass (October 31, 1815-February 19, 1897), sometimes called the "father of modern analysis," did research in some of the topics covered in our Analysis 1 (MATH 4217/5217) and Analysis 2 (MATH 4227/5227) classes, but is also remembered as a influential math instructor. From 1856 to 1890 he taught at the University of Berlin. In his lectures, he sought to place calculus and analysis on a rigorous foundation. He taught classes on introduction to analysis, integral calculus, and the general theory of analytic functions, and made contributions to the calculus of variations. In 1872 he discovered the continuous no-where-differentiable function mentioned in Exercise 3.2.64.

These comments are based on the Weierstrass biographies on Wikipedia and the MacTutor History of Mathematics Archive. The image above is from the MacTutor History of Mathematics Archive.

Examples. Exercise 3.2.50 and Exercise 3.2.56.

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