Chapter 3. Derivatives

3.4. The Derivative as a Rate of Change

Note. In this section we use derivatives to measure the rate at which some quantity (measured by a function $f(x)$) changes as the input variable x changes. This is why calculus is so useful in physics applications, where you consider position as a function of time so that the derivative represents velocity and the second derivative represents acceleration.

Definition. Instantaneous Rate of Change.
The *instantaneous rate of change* of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists.

Definition. If $s = f(t)$ represents the location of an object lying along a coordinate axis (for applications, we may take the axis as a horizontal line or as a vertical line), then the *displacement* of the object over the time from t to $t + \Delta t$ is $\Delta s = f(t + \Delta t) - f(t)$, and the *average velocity* of the object over that time interval is

$$v_{av} = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

Note. If the coordinate axis is a horizontal line, then we might have the displacement as given in Figure 3.15. Notice that in this figure the displacement is positive since $f(t + \Delta t) > f(t)$ (so the object has moved to the right over time Δt).

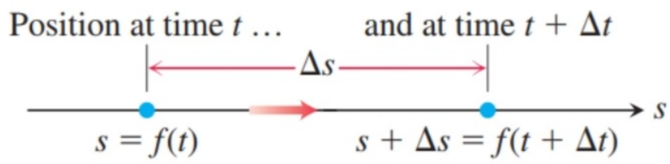


Figure 3.15

Definition. (Instantaneous) Velocity.

Velocity (*instantaneous velocity*) is the derivative of position with respect to time. If a body's position at time t is $s = f(t)$, then the body's velocity at time t is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

Note. If $s = f(t)$ represents height so that the coordinate axis is vertical, then when $ds/dt > 0$ we have that the object is moving upward and when $ds/dt < 0$ we have that the object is moving downward:

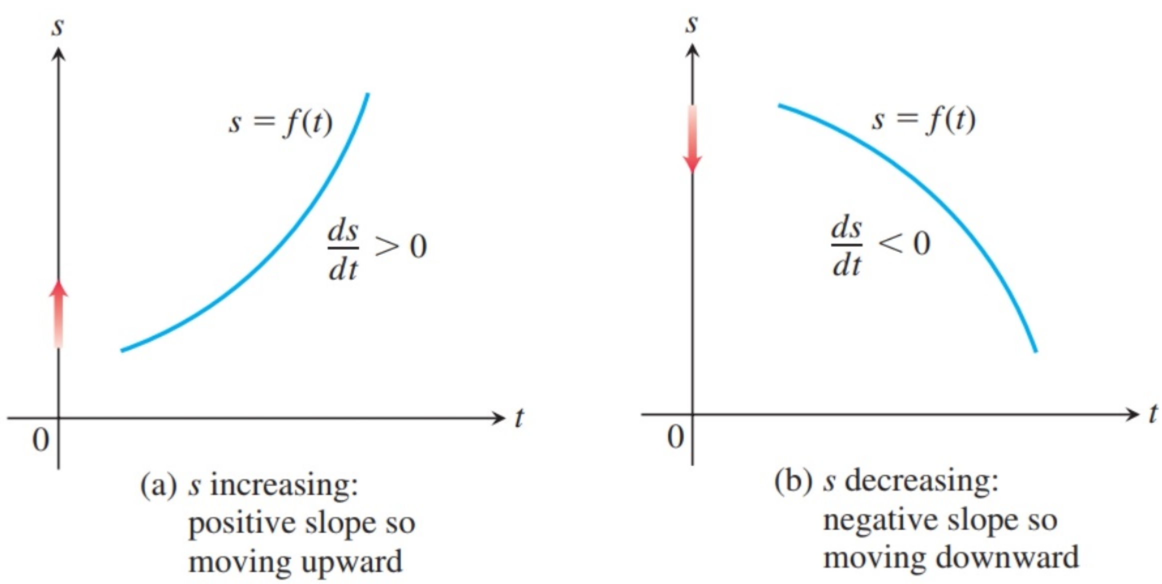


Figure 3.16

Definition. Speed

Speed is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|.$$

Note. Notice that speed is nonnegative. You may have hear it said that "velocity is a vector quantity" and "speed is a scalar quantity." In our setting which deals with motion along a line (called "rectilinear motion"), we indicate the direction of velocity (or acceleration) by the presence of a positive sign for movement in the positive direction (i.e., to the right for a horizontal coordinate axis and up for a vertical coordinate axis), or by the presence of a negative sign for movement in the negative direction (i.e., to the left for a horizontal coordinate axis and down for a vertical coordinate axis). Speed is then the magnitude of velocity.

Example. Example 3.4.2.

Definition. Acceleration and Jerk.

Acceleration is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

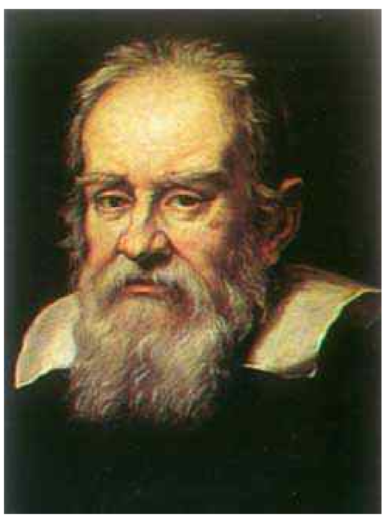
$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Jerk is the derivative of acceleration with respect to time:

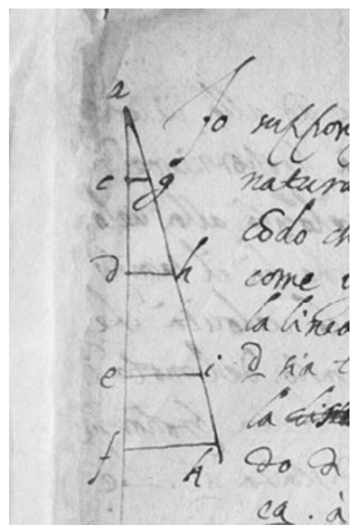
$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

Example. Exercise 3.4.12.

Note. In exploring the velocity and acceleration of an object experimentally, Galileo Galilei (1564–1642) rolled balls down an inclined ramp. His plan was to have an object fall at an angle so that the speed of fall would be slower than when the object was in free fall. In this way, he could better measure the time it took the object to move a certain distance (he measured time using a pendulum clock; of course he didn't have a stopwatch!). His experiments are describe in the online video NOVA: Galileo's Inclined Plane, available on PBS LearningMedia site of East Tennessee PBS (accessed 7/25/2020). Exercise 3.4.14 deals with this experiment.



Galileo Galilei



Galileo's writing of 1604 where he derives the times-squared law

The Galileo image is from MacTutor History of Mathematics Archive and the writing is from Paoloa Palmieri's "Galileo's Construction of Idealized Fall in the Void," *History of Science* **43**, 343–389 (2005); available online on Palmieri's webpage. It is rumored that Galileo did a similar experiment by dropping objects off of the

Leaning Tower of Pisa. This story is from a biography of Galileo by one of his students, Vincenzo Viviani, written in 1654 and published in 1717. However, this story is thought to be questionable; see the Wikipedia page on Galileo's Leaning Tower of Pisa experiment. Exercise 3.4.13 deals with this story (notice that the question reads "*Had* Galileo dropped a cannonball from the Tower of Pisa...," my emphasis).

Example. Exercise 3.4.14.

Note. At the surface of the Earth, if an object is fired directly upward with an initial (upward) velocity v_0 from an initial height s_0, then the height of the object at time t is

$$s(t) = -16t^2 + v_0 t + s_0$$

if time is measured in seconds and distances are measured in feet, or

$$s(t) = -4.9t^2 + v_0 t + s_0$$

if time is measured in seconds and distances are measured in meters. Notice that the acceleration is $s''(t) = \frac{d}{dt}[s'(t)] = \frac{d}{dt}[-32t + v_0] = -32$ ft/sec^2, or $s''(t) = \frac{d}{dt}[s'(t)] = \frac{d}{dt}[-9.8t + v_0] = -9.8$ m/sec^2.

Example. Example 3.3.4.

Example. Exercise 3.4.22.

Note. In economics, the term "*marginal*" is used when referring to derivatives. If a company produces and sells a number x of objects, and the cost of producing those objects is $c(x)$ and the revenue that results from selling them is $r(x)$, then the resulting profit is $p(x) = r(x) - c(x)$. The functions $p'(x)$, $r'(x)$, and $c'(x)$ are the marginal profit, revenue, and cost functions, respectively.

Examples. Exercise 3.4.24 and Exercise 3.4.30.

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