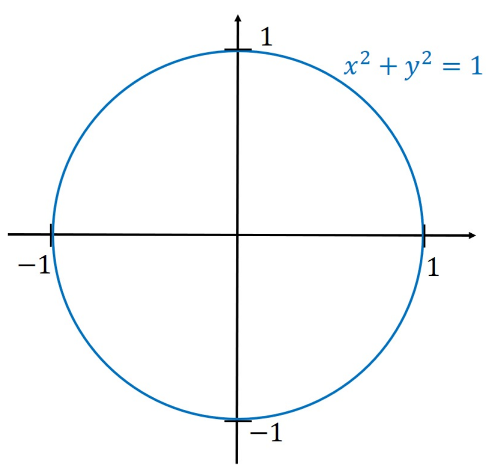
Chapter 3. Derivatives

3.7. Implicit Differentiation

Note. In this section we define what it means for a *function* to be implicit to an *equation* and we give a process by which we can compute derivatives of such functions.

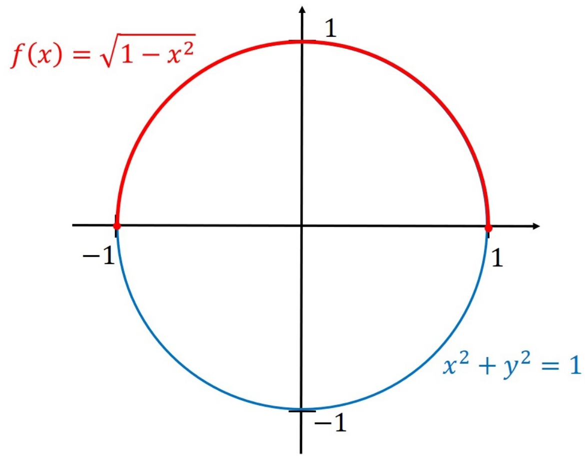
Definition. The function $f(x)$ is *implicit* to the equation $F(x, y) = 0$ if the substitution $y = f(x)$ into the equation yields an identity.

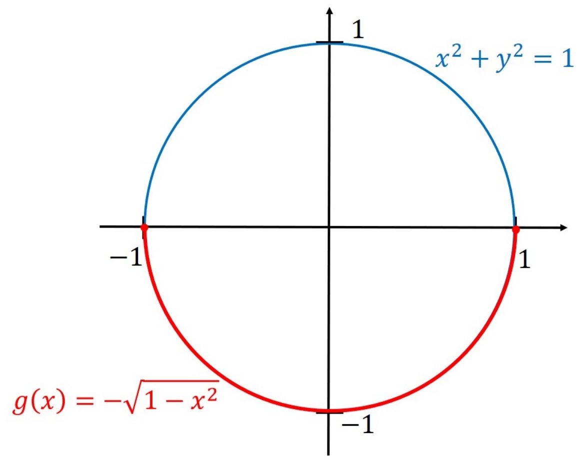
Example. Consider the equation $x^2 + y^2 = 1$, which has as its graph the unit circle centered at the origin:



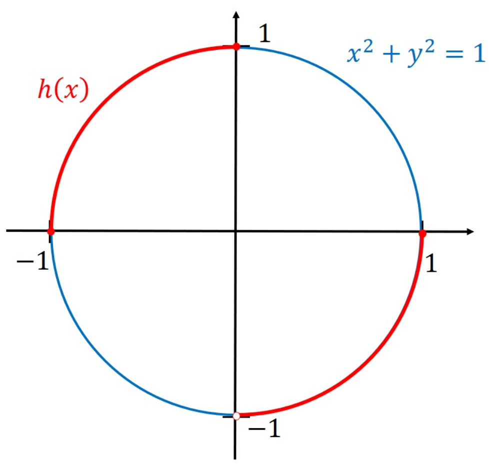
The function $f(x) = \sqrt{1 - x^2}$ is implicit to the equation $x^2 + y^2 = 1$, since with $y = f(x)$ we have $x^2 + y^2 = x^2 + (f(x))^2 = x^2 + \left(\sqrt{1 - x^2}\right)^2 = x^2 + (1 - x^2) = 1$. The function $g(x) = -\sqrt{1 - x^2}$ is implicit to the equation $x^2 + y^2 = 1$, since with $y = g(x)$ we have $x^2 + y^2 = x^2 + (g(x))^2 = x^2 + \left(-\sqrt{1 - x^2}\right)^2 = x^2 + (1 - x^2) = 1$.

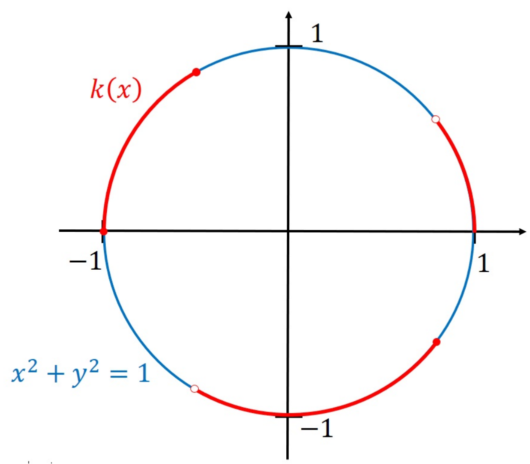
Notice the graphs of these two functions have domain $[-1, 1]$ and their graphs coincide with part of the graph of $x^2 + y^2 = 1$:





These are the only *continuous* functions defined on $[-1, 1]$ which are implicit to $x^2 + y^2 = 1$. However, there are infinitely many discontinuous functions implicit to the equation. Here are two such functions:





Note. If $y = f(x)$ is a function implicit to $F(x, y) = 0$, then we can generate an equation containing dy/dx by differentiating "implicitly." This follows by applying the Chain Rule. Since a function is not (necessarily) uniquely determined by an equation, we may not get an explicit formula for dy/dx in terms of x values only.

Example. Suppose $y = f(x)$ is implicit to $x^2 + y^2 = 1$. Then differentiating implicitly:

$$\begin{aligned}
\frac{d}{dx}[x^2 + y^2] &= \frac{d}{dx}[1] \\
\frac{d}{dx}[x^2] + \frac{d}{dx}[y^2] &= \frac{d}{dx}[1] \\
\frac{d}{dx}[x^2] + \frac{d}{dx}[(f(x))^2] &= \frac{d}{dx}[1] \\
2x + 2f(x)\overset{\curvearrowright}{[}f'(x)] &= 0 \\
2x + 2y\overset{\curvearrowright}{\left[\frac{dy}{dx}\right]} &= 0 \\
2y\frac{dy}{dx} &= -2x \\
\frac{dy}{dx} &= -\frac{x}{y}.
\end{aligned}$$

Notice that dy/dx involves both x and y. This is because we cannot find the slope of a line tangent to the graph of $F(x, y) = 0$ without knowing the x and y coordinates of the point of tangency. That is, for a given x value there are multiple corresponding y values (exactly two such y values for each $x \in (-1, 1)$).

Example 3.7.A. Find the slope of the line tangent to $x^2 + y^2 = 1$ at $(x, y) = (\sqrt{2}/2, \sqrt{2}/2)$. Do the same for the point $(x, y) = (\sqrt{2}/2, -\sqrt{2}/2)$.

Examples. Exercise 3.7.16 and Exercise 3.7.20.

Definition. A line is *normal* to a curve at a point if it is perpendicular to the curve's tangent line. The line is called the *normal* to the curve at the point.

Note. When light enters a lens, the angle that the ray of light makes with the normal line to the lens (angle A in Figure 3.33) and the angle the ray of light makes with the normal line to the lens once the ray is inside the lens (angle B in Figure 3.33) are related by *Snell's Law* which state $n_A \sin A = n_B \sin B$ where n_A is the refractive index of the medium outside the lens (presumably air) and n_B is the refractive index of the medium out of which the lens is made (presumably glass). So there are physical reasons to have an interest in the normal line to a curve (the curve here being a profile curve of the lens).

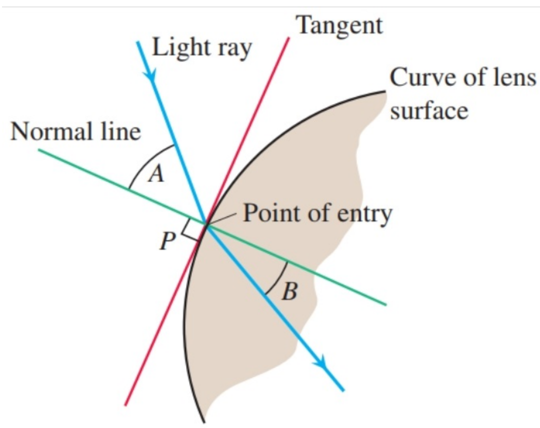


Figure 3.33

Examples. Exercise 3.7.40 and Exercise 3.7.44.

Note. Just as we can use the Chain Rule to find the derivative of a function implicit to an equation, we can also use it to find second (and higher) order derivatives of implicit functions.

Example. Exercise 3.7.22.

Examples. Exercise 3.7.48 and Exercise 3.7.50.