Chapter 3. Derivatives

3.8. Derivatives of Inverse Functions and Logarithms

Note. In this section we explore the relationship between the derivative of an invertible function and the derivative of its inverse. This leads us to consider derivatives of logarithmic and exponential functions.

Note. Recall that the graph of a one-to-one function f and its inverse f^{-1} are mirror images of each other about the line $y = x$. In Figure 3.37 we see the graph of the one-to-one function $f(x) = x^2$, $x \geq 0$, and its inverse $f^{-1}(x) = \sqrt{x}$. Notice that the points $(4, 2)$ and $(2, 4)$ are mirror images of each other about the line $y = x$; the slope of $y = f(x) = x^2$, $x \geq 0$ at $(2, 4)$ is 4 and the slope of $y = f^{-1}(x) = \sqrt{x}$ at $(4, 2)$ is $1/4$. This reciprocal relationship is not a coincidence.

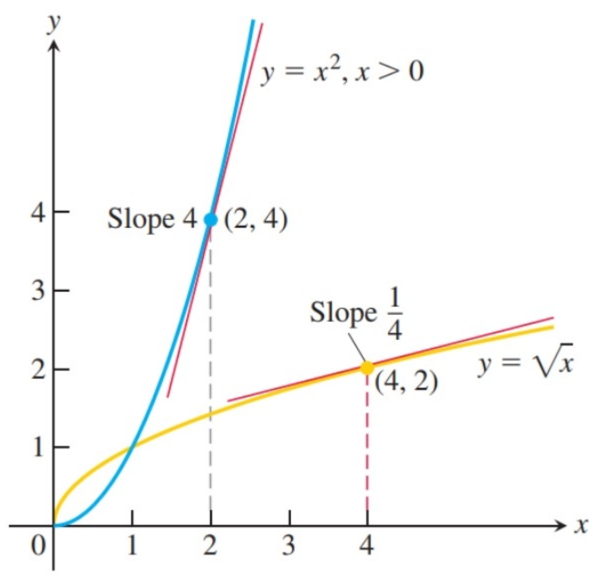

Figure 3.37

Theorem 3.3. The Derivative Rule for Inverses

If f has an interval I as its domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain. The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}.$$

Example. Exercise 3.8.8.

Note. Since we can differentiate e^x and $\ln x$ is the inverse of e^x, then we can use Theorem 3.3 to differentiate $\ln x$.

Theorem 3.8.A. For $x > 0$ we have

$$\frac{d}{dx}[\ln x] = \frac{1}{x}.$$

If $u = u(x)$ is a differentiable function of x, then for all x such that $u(x) > 0$ we have

$$\frac{d}{dx}[\ln u] = \frac{d}{dx}[\ln u(x)] = \frac{1}{u}\left[\overset{\curvearrowright}{\frac{du}{dx}}\right] = \frac{1}{u(x)}\overset{\curvearrowright}{[u'(x)]}.$$

Note. We can apply the previous theorem to show that $\frac{d}{dx}[\ln|x|] = \frac{1}{x}$ for $x \neq 0$ (see Example 3.8.3(c)).

Examples. Exercise 3.8.16, Exercise 3.8.30, and Exercise 3.8.38.

Note. The previous example suggests that the computation of certain derivatives (those involving lots of products and quotients, or raising to powers) can be simplified by first taking a natural logarithm. This technique is called *logarithmic differentiation* and requires the use of the Chain Rule (Theorem 3.2). We illustrate it with an example.

Example. Exercise 3.8.52: Find y' by first taking a natural logarithm and then differentiating implicitly: $y = \sqrt{\dfrac{(x+1)^{10}}{(2x+1)^5}}$.

Note. When $a \in \mathbb{R}$, by a^n where n is a positive integer, we mean $(a)(a)\cdots(a)$ (n times). When $-m$ is a negative integer, by a^{-m} we mean $1/a^m = (1/a)(1/a)\cdots(1/a)$ (m times). For m/n a rational number, by $a^{m/n}$ we mean $\sqrt[n]{a^m}$ (provided this is defined and we avoid even roots of negative numbers). So this takes care of defining a^r for r an integer or rational number (provided $a > 0$ or $a < 0$ and we avoid the even roots of negatives problem). Now what if r is irrational? We now use the natural exponential function to define what it means to raise a positive real number to any real number power, including irrational powers.

Definition. For any numbers $a > 0$ and for any real x, $a^x = e^{x\ln a}$.

Note. Now that we have introduced a new function, a^x, we want to differentiate it.

Theorem 3.8.B. If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx}[a^u] = (\ln a)\overset{\curvearrowright}{a^u}\left[\frac{du}{dx}\right].$$

Note. Notice that the previous theorem implies that $\frac{d}{dx}[a^x] = a^x \ln a$. With $a = e$, we have the special case $\frac{d}{dx}[e^x] = e^x(1) = e^x$. Again, this is what is *natural* about e. When you first meet the natural exponential and logarithmic functions in algebra, it is hard to understand what is NATURAL about them. That is because the "natural-ness" is a calculus property (namely this differentiation property).

Note. We saw in Section 3.3 that $\frac{d}{dx}[a^x] = a^x \left(\lim_{h \to 0} \frac{a^h - 1}{h} \right)$. We said then that the limit exists. We now see that the limit is $\lim_{h \to 0} \frac{a^h - 1}{h} = \ln a = L_a$. In particular, for $a = e$, $\lim_{h \to 0} \frac{e^h - 1}{h} = \ln e = 1$.

Example. Exercise 3.8.70.

Definition. For any $a > 0$, $a \neq 1$, define $\log_a x = \frac{\ln x}{\ln a}$. (This is called the *change of base* formula. See Section 1.6.)

Note. Now that we have introduced a another new function, $\log_a x$, we want to differentiate it.

Theorem 3.8.C. Differentiating a logarithm base a gives:

$$\frac{d}{dx}[\log_a u] = \frac{1}{\ln a}\overset{\curvearrowright}{\frac{1}{u}}\left[\frac{du}{dx}\right].$$

Examples. Exercise 3.8.74 and Exercise 3.8.80.

Example. Exercise 3.8.90: Use logarithmic differentiation to find dy/dx: $y = x^{x+1}$.

Definition. For any $x > 0$ and for any real number n, define $x^n = e^{n \ln x}$.

Note. From the definition of a^x, where $a > 0$, as $a^x = e^{x \ln a}$, we see that the previous definition follows by taking $a = x$ and $n = x$ in $a^x = e^{x \ln a}$. We can now prove the General Power Rule for Derivatives (Theorem 3.3.C) from Section 3.3.

Theorem 3.3.C/3.8.D. General Power Rule for Derivatives.
For $x > 0$ and any real number n,

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

If $x < 0$, then the formula holds whenever the derivative, x^n, and x^{n-1} all exist.

Example. Example 3.8.72: Differentiate $y = t^{1-e}$.

Note. We gave an approximation of the irrational number e in Section 3.3 of $e \approx 2.718281828459045 9$. In the next theorem we give an exact value of e... but we give it as a limit.

Theorem 3.4. The Number e as a Limit

We can find e as a limit:

$$e = \lim_{x \to 0} (1 + x)^{1/x}.$$

Note. By computing $(1+x)^{1/x}$ for "really small" values of x, we can get a decimal approximation of e, as stated above.

Example. Exercise 3.8.102.

Revised: 8/13/2020