Chapter 3. Derivatives

3.9. Inverse Trigonometric Functions

Note. In this section we review the definitions of the inverse trigonometric functions from Section 1.6. Inverse Functions and Logarithms. We then apply the same technique used to prove Theorem 3.3, "The Derivative Rule for Inverses," to differentiate each inverse trigonometric function.

Note. Recall from Section 1.6 that the six inverse trigonometric functions are defined as follows:

1. $y = \cos^{-1} x$ if and only if $\cos y = x$ and $y \in [0, \pi]$.

2. $y = \sin^{-1} x$ if and only if $\sin y = x$ and $y \in [-\pi/2, \pi/2]$.

3. $y = \tan^{-1} x$ if and only if $\tan y = x$ and $y \in (-\pi/2, \pi/2)$.

4. $y = \sec^{-1} x$ if and only if $\sec y = x$ and $y \in [0, \pi/2) \bigcup (\pi/2, \pi]$.

5. $y = \csc^{-1} x$ if and only if $\csc y = x$ and $y \in [-\pi/2, 0) \bigcup (0, \pi/2]$.

6. $y = \cot^{-1} x$ if and only if $\cot y = x$ and $y \in (0, \pi)$.

For all appropriate x values:

$$\begin{aligned} \sec^{-1} x &= \pi - \csc^{-1} x = \pi/2 - \sin^{-1}(1/x) = \cos^{-1}(1/x) \\ \csc^{-1} x &= \pi/2 - \sec^{-1} x = \pi/2 - \cos^{1} 1/x = \sin^{-1}(1/x) \\ \cot^{-1} x &= \pi/2 - \tan^{-1} x. \end{aligned}$$

Note. The text book cautions that the choice of $\sec^{-1} x$ for x negative is not universal. The choice that $\pi/2 < \sec^{-1} x \leq \pi$ for x negative (namely, $x \leq -1$) implies the relationship $\sec^{-1} x = \cos^{-1}(1/x)$, as stated above. Other sources choose $-\pi \leq \sec^{-1} x < -\pi/2$ and others choose $\pi \leq \sec^{-1} x < 3\pi/2$ for x negative (see Figure 3.42). This just reflects the difficulty of finding an inverse of a non-one-to-one function.

Note. The graphs of the six inverse trig functions are:

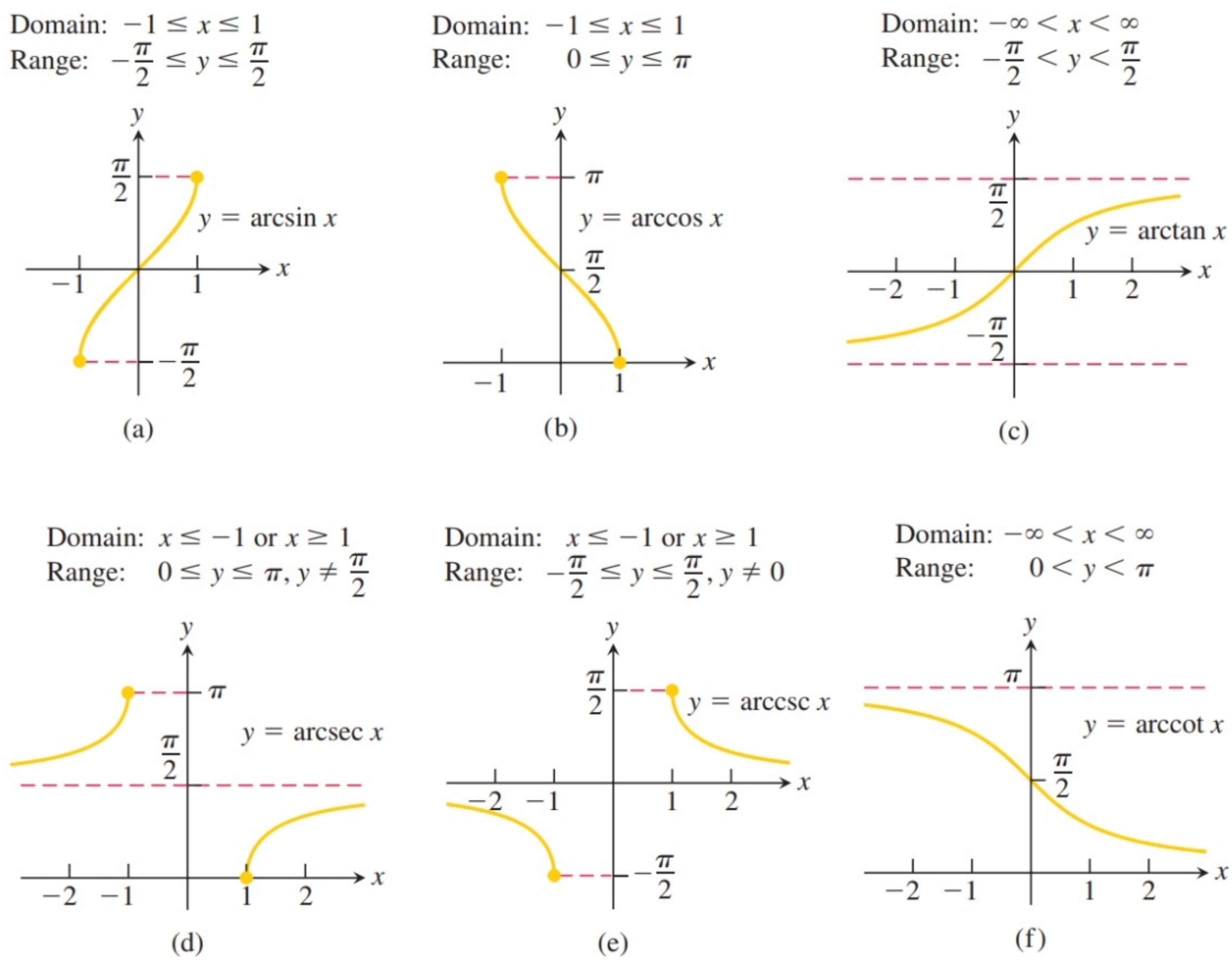

Figure 1.64

Examples. Exercise 3.9.4 and Exercise 3.9.14.

Theorem 3.9.A. We differentiate $\sin^{-1}$ as follows:

$$\frac{d}{dx}\left[\sin^{-1} u\right] = \frac{1}{\sqrt{1-u^2}}\,\overset{\curvearrowright}{\left[\frac{du}{dx}\right]}$$

where $|u| < 1$.

Example. Exercise 3.9.24.

Theorem 3.9.B. We differentiate $\tan^{-1}$ as follows:

$$\frac{d}{dx}\left[\tan^{-1} u\right] = \frac{1}{1+u^2}\,\overset{\curvearrowright}{\left[\frac{du}{dx}\right]}.$$

Example. Exercise 3.9.34.

Theorem 3.9.C. We differentiate $\sec^{-1}$ as follows:

$$\frac{d}{dx}\left[\sec^{-1} u\right] = \frac{1}{|u|\sqrt{u^2-1}}\,\overset{\curvearrowright}{\left[\frac{du}{dx}\right]}$$

where $|u| > 1$.

Note. We can use the following identities to differentiate the other three inverse trig functions:

$$\begin{aligned}
\cos^{-1} x &= \pi/2 - \sin^{-1} x \\
\cot^{-1} x &= \pi/2 - \tan^{-1} x \\
\csc^{-1} x &= \pi/2 - \sec^{-1} x
\end{aligned}$$

We then see that the only difference in the derivative of an inverse trig function and the derivative of the inverse of its cofunction is a negative sign. In summary, that is (Table 3.1):

1. $\dfrac{d}{dx}\left[\sin^{-1} u\right] = \dfrac{du/dx}{\sqrt{1-u^2}}, |u| < 1$
2. $\dfrac{d}{dx}\left[\cos^{-1} u\right] = -\dfrac{du/dx}{\sqrt{1-u^2}}, |u| < 1$
3. $\dfrac{d}{dx}\left[\tan^{-1} u\right] = \dfrac{du/dx}{1+u^2}$
4. $\dfrac{d}{dx}\left[\cot^{-1} u\right] = -\dfrac{du/dx}{1+u^2}$
5. $\dfrac{d}{dx}\left[\sec^{-1} u\right] = \dfrac{du/dx}{|u|\sqrt{u^2-1}}, |u| > 1$
6. $\dfrac{d}{dx}\left[\csc^{-1} u\right] = \dfrac{-du/dx}{|u|\sqrt{u^2-1}}, |u| < 1$

Examples. Exercise 3.9.40, Exercise 3.9.44, and Exercise 3.9.60.

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