Chapter 4. Applications of Derivatives

Note. In this chapter, we use derivatives to find extrema of functions, to graph functions using both the first and second derivative, find extrema in applied problems, and introduce a technique which reverses differentiation.

4.1. Extreme Values of Functions on Closed Intervals

Note. In this section use the derivative of a function to find local and absolute maximums and minimums.

Definition. Let f be a function with domain D. Then f has

(a) an *absolute maximum* value on D at a point c if $f(x) \leq f(c)$ for all x in D, and

(b) an *absolute minimum* value on D at point c if $f(x) \geq f(c)$ for all x in D.

Maximum and minimum values are *extreme values* of the function f. Absolute maxima or minima are also called *global* maxima and minima.

Example 4.1.1. A function may or may not have absolute extrema. It is dependent on the function and the domain. In Figure 4.2, there are examples of functions with (a) an absolute minimum only, (b) both and absolute maximum and minimum, (c) an absolute maximum only, and (d) neither an absolute maximum nor absolute minimum.

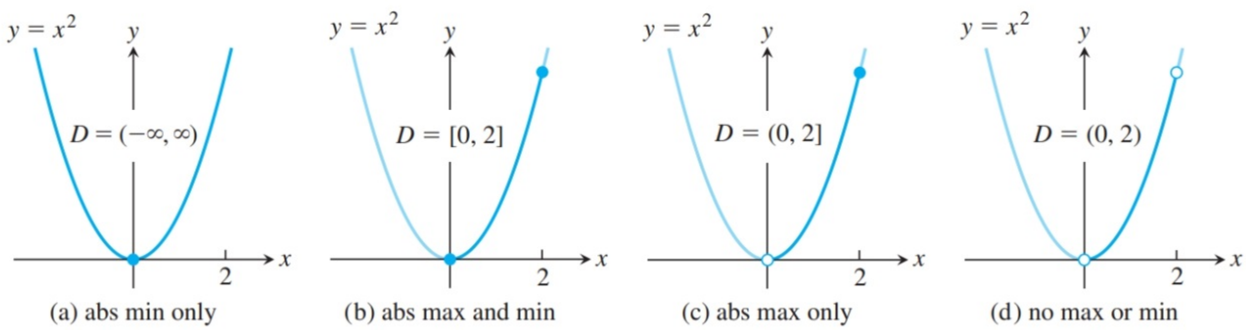

Figure 4.2

Note. The next result gives a condition under which a function has extrema on a set of real numbers. A proof of the result is involved and requires several deep ideas. See my online notes for Analysis 1 (MATH 4217/5217) on 4.1 Limits and Continuity; see Corollary 4-7(b), The Extreme Value Theorem.

Theorem 4.1. The Extreme-Value Theorem for Continuous Functions
If f is continuous at every point of a closed and bounded interval $I = [a, b]$, then f assumes both an absolute maximum value M and an absolute minimum value m somewhere in I. That is, there are numbers x_1 and x_2 in $I = [a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every x in $I = [a, b]$.

Examples. Exercise 4.1.2 and Exercise 4.1.4.

Definition. Let c be an interior point of the domain of the function f. Then $f(c)$ is a

(a) *local maximum value* if and only if $f(x) \leq f(c)$ for all x in some open interval containing c

(b) *local minimum value* if and only if $f(x) \geq f(c)$ for all x in some open interval containing c.

Local extrema are also called *relative extrema*.

Note. In Figure 4.5 we see a function with several local extrema, and see how a local extrema can occur at an endpoint of a domain.

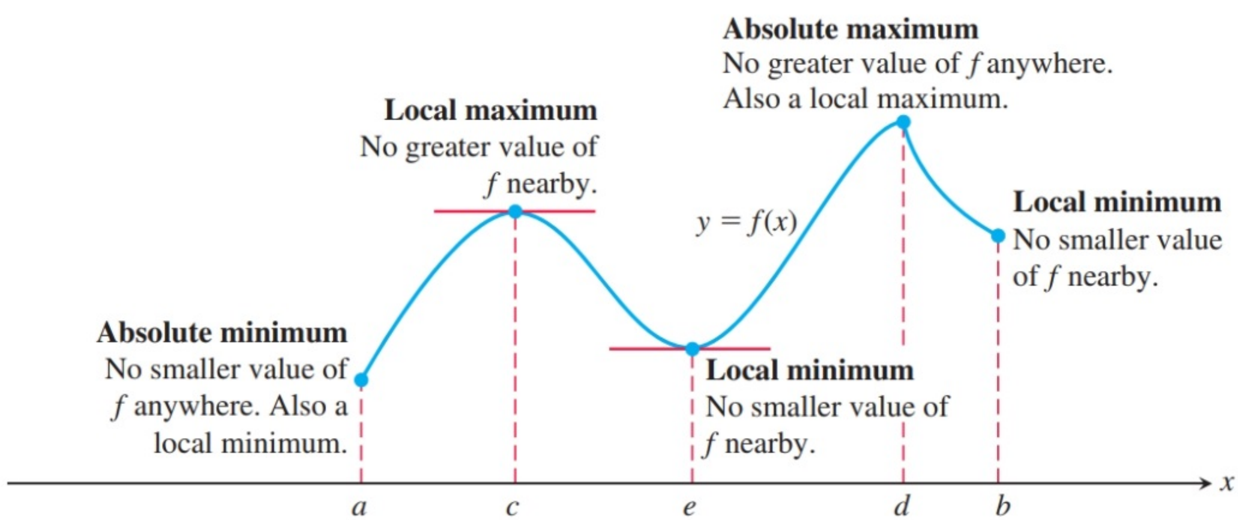

Figure 4.5

Note. We now state and prove a result that tells us where to look for local extrema of a differentiable function.

Theorem 4.2. Local Extreme Values.

If a function f has a local maximum value or a local minimum value at an interior point c of its domain, and if f' exists at c, then $f'(c) = 0$.

Note. Theorem 4.2 implies that for a function f with a domain which is an interval (or a union of intervals), the local extreme values occur at:

1. interior points where f' is 0,

2. interior points were f' is undefined, or

3. endpoints of the domain of f.

We give the types of points where f' is 0 or undefined a special name, as follows.

Definition. A point in the domain of a function f at which f' is 0 or f' does not exist is a *critical point* of f.

Note. To find extrema on a closed and bounded interval we:

Step 1. Find all critical points of f on the interval.

Step 2. Evaluate f at all critical points and endpoints.

Step 3. Take the largest and smallest of these values.

Examples. Exercise 4.1.24, Exercise 4.1.44, Exercise 4.1.60, and Exercise 4.1.72.

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